

The intuitionistic temporal logic of dynamical systems

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Abstract

A *dynamical system* is a pair (X, f) , where X is a topological space and $f: X \rightarrow X$ is continuous. Kremer observed that the language of propositional linear temporal logic can be interpreted over the class of dynamical systems, giving rise to a natural intuitionistic temporal logic. We introduce a variant of Kremer's logic, which we call ITLC, and show that it is decidable.

1 Introduction

What is intuitionistic temporal logic? This is an interesting question that does not necessarily have a unique answer, but there are certainly some natural candidates. To be precise, we are interested in intuitionistic analogues of propositional linear temporal logic LTL (see [18] for an overview). Ewald considered intuitionistic logic with 'past' and 'future' tenses [7] and Davies suggested an intuitionistic temporal logic with 'next' [5], later studied by Kojima and Igarashi [12], who endowed it with Kripke semantics and provided a complete deductive system. Maier observed that an intuitionistic temporal logic with infinitary operators including 'henceforth' could be used for reasoning about safety and liveness conditions in possibly-terminating reactive systems [19]. Logics with henceforth were later studied by Kamide and Wansing [11], albeit in bounded time.

Each of [7, 12, 11] use semantics based on bi-relational models for intuitionistic modal logic, studied systematically by Simpson [22]. In the presence of next,

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henceforth, and unbounded time, there are many open questions regarding the decidability, or even axiomatizability, of intuitionistic LTL, for example with respect to the class of functional Kripke frames from [12]. Nevertheless, there are some related logics which have been axiomatized: Nishimura [20] provided a sound and complete axiomatization for an intuitionistic variant of propositional dynamic logic PDL, using a different class of bi-relational frames. More recently, Balbiani and Diéguez axiomatized the ‘here-and-there’ variant of LTL [3].

In this work we follow an entirely different approach. Intuitionistic modal logic can also be interpreted topologically, as observed by Davoren et al. [6], and in particular, Kremer suggested that intuitionistic LTL can be interpreted over dynamical systems [15]. The latter interpretation follows the tradition of dynamic topological logic, which originates from work of Artemov et al. [2]. There, they proposed a propositional logical framework for reasoning about *hybrid systems*; these are structures $\mathfrak{X} = (X, \mathcal{T}, f)$, where (X, \mathcal{T}) is a topological space and $f: X \rightarrow X$ is any function. In the case that f is continuous, we will say that \mathfrak{X} is a *dynamical system*. Hybrid systems are ubiquitous in e.g. physics, biology and computer science, as they may be used to model situations where continuous quantities are modified over discrete time, such as a computation with real numbers. The logics proposed by Artemov et al. include one modality, which we will denote \blacksquare , interpreted as an interior operator in the sense of Tarski [23], and a temporal-like modality, which we will denote \circ , interpreted using the function f . They introduced two such logics, S4F and S4C, and proved that both of them are decidable, as well as sound and complete for the class, respectively, of all hybrid systems, or of all dynamical systems.

Kremer and Mints [17] considered a similar logic, called S4H, and also showed it to be sound and complete for dynamical systems where f is a homeomorphism. They also observed that adding a ‘henceforth’ operator, \Box , would potentially combine the advantages of S4F and S4C with those of LTL, thus allowing us to express and reason about the asymptotic properties of hybrid systems, including recurrence phenomena. The resulting tri-modal system was called *dynamic topological logic*. Unfortunately, it was soon shown by Konev et al. that dynamic topological logic is undecidable over the class of all dynamical systems [13]. Later we provided a sound and complete axiomatization [10], but Konev et al. showed that DTL over the class of dynamical systems with a homeomorphism is not even computably enumerable [14].

This led to a search for variants of DTL which retained the capacity for reasoning about asymptotic behavior but remained decidable. Gabelaia et al. proposed to consider dynamic topological logics with finite, but unbounded, time and showed them to be decidable, although not in primitive recursive time. Kremer instead proposed a restriction to dynamical systems where the topology is a partition [16], while we considered interpretations over minimal systems¹ [9]; the DTL’s obtained in both of these cases are decidable.

However, as a general rule, all of the decidable variants of DTL with contin-

¹*Minimal systems*, introduced in [4], are dynamical systems where the orbit of every point is dense; equivalently, they are dynamical systems that admit no non-empty, closed, f -invariant, proper subsystems.

uous functions that are currently known are obtained by either restricting the class of dynamical systems over which they are interpreted, or restricting the logics to reason about finitely many iterations of f .² However, there is another variation of DTL, which does not have either of these restrictions, yet whose decidability was never settled; Kremer’s intuitionistic version of DTL, proposed in unpublished work [15]. It is well-known that propositional intuitionistic logic can be seen as a fragment of **S4** via the Gödel-Tarski translation [23], and indeed the two share very similar semantics. In particular, intuitionistic logic can be interpreted topologically. One can use this idea to present a version of dynamic topological logic which removes the modality \blacksquare , and instead interprets Boolean connectives intuitionistically. As Kremer observed, when f is a continuous function, the truth condition for $\circ\varphi$ will preserve the openness of the truth valuation.

On the other hand, Kremer also observed that the classical truth condition for $\Box\varphi$ did not always produce an open set, and thus he proposed to take the interior of this classical interpretation (see §3 for details). He then showed that several key principles of LTL, namely $\Box p \rightarrow \circ\Box p$, $\Box \circ p \rightarrow \circ\Box p$, and $\Box p \rightarrow \Box\Box p$, are not valid for his semantics.

We will follow Kremer in using intuitionistic temporal logic to reason about dynamical systems, but we will replace \Box by \Diamond ; note that the two are not inter-definable intuitionistically. Working with \Diamond is convenient because the classical interpretation of $\Diamond\varphi$ will indeed yield open sets. We will also add a universal modality, and call the resulting logic ITLC (*Intuitionistic Temporal Logic of Continuous functions*); our main result is that ITLC is decidable. We prove this by showing that, while ITLC does not have the finite model property, it does have an effective *quasimodel* property.

We use techniques based on non-deterministic quasimodels, first introduced in [8]. There are a few simplifications due to the intuitionistic semantics, which require us to work only with open sets, as well as partial orders rather than preorders when dealing with relational semantics. Moreover, we reduced the semantics of DTL to countable quasimodels, but those of ITLC can be reduced to *finite* quasimodels; this is, of course, good news, but requires new combinatorial work with what we call irreducible moments.

Layout

The layout of the article is as follows.

- II We briefly review some basic notions including topology and its relation to partial orders.
- III We introduce the logic ITLC and its intended semantics using dynamical models.
- IV We compare our logic ITLC to some related logics found in the literature.

²An exception to this is the ‘temporal over topological’ fragment discussed in [17], but this fragment is too weak to express continuity.

- V We introduce labeled spaces and systems, which generalize both dynamical models and quasimodels, defined as a special case.
- VI We show that any formula falsifiable in a quasimodel is falsifiable in a dynamical model.
- VII The rest of the article is devoted to showing the opposite implication. To this end, we introduce moments, which are essentially the points in our ‘weak canonical quasimodel’ \mathbb{M}_Σ .
- VIII We discuss simulations, the essential tool for extracting quasimodels out of dynamical models.
- IX Since the structure \mathbb{M}_Σ is infinite, we introduce the finite substructure of irreducible moments, denoted \mathbb{I}_Σ .
- X Finally, we use all of the tools developed in previous sections to show that a formula is valid if and only if it is valid over an effectively bounded quasimodel.

2 Binary Relations and Topology

In this section we establish some of the notation we will use and recall some basic notions from topology. In particular, we discuss Aleksandroff spaces, which link partial orders and topological spaces.

2.1 Sets and relations

We follow fairly standard conventions for sets and relations. The first infinite ordinal is denoted ω , the cardinality of a set A will be denoted $\#A$, and its powerset will be denoted $\mathcal{P}(A)$. If $R \subseteq A \times B$ and $X \subseteq A$, we write $R(X)$, or simply RX , for $\{y \in B : \exists x \in X \ x R y\}$, and we write R^{-1} for $\{(y, x) \in B \times A : (x, y) \in R\}$. We write $\text{dom}(R)$ for $R^{-1}(B)$ and $\text{rng}(R)$ for $R(A)$. R is *total* if $\text{dom}(R) = A$, and *surjective* if $\text{rng}(R) = B$. We define the restriction of R to X by $R \upharpoonright X = R \cap (X \times B)$, except when $A = B$, in which case $R \upharpoonright X = R \cap (X \times X)$. If, moreover, $S \subseteq B \times C$, the composition of R and S is $S \circ R \subseteq A \times C$. We may simply write SR instead of $S \circ R$.

If $B = A$ (so that $R \subseteq A \times A$), recall that R is (a) *reflexive* if for every $x \in A$, $x R x$; (b) *antisymmetric* if for every $x, y \in A$, if $x R y$ and $y R x$, then $x = y$; (c) *transitive* if, whenever $x R y$ and $y R z$, it follows that $x R z$; (d) *linear* if, for all $x, y \in A$, either $x \preceq y$ or $y \preceq x$, and (e) *serial*, if for every $x \in A$ there is $y \in A$ such that $x R y$.

A transitive, reflexive relation is a *preorder* and a transitive, reflexive, antisymmetric relation is a *partial order*. A *poset* is a pair $\mathfrak{A} = (|\mathfrak{A}|, \preceq_\mathfrak{A})$, where $|\mathfrak{A}|$ is any set and $\preceq_\mathfrak{A} \subseteq |\mathfrak{A}| \times |\mathfrak{A}|$ is a partial order; we will generally adopt the convention of denoting the domain of a structure \mathfrak{A} by $|\mathfrak{A}|$, and if $X \subseteq |\mathfrak{A}|$, $\mathfrak{A} \upharpoonright X$ is the structure obtained by restricting each element of \mathfrak{A} to X . Define

$\downarrow_{\mathfrak{A}} a = \{b : b \preceq a\}$ and $\uparrow_{\mathfrak{A}} a = \{b : b \succeq a\}$. We will generally write $\preceq, \downarrow, \uparrow$ instead of $\preceq_{\mathfrak{A}}, \downarrow_{\mathfrak{A}}, \uparrow_{\mathfrak{A}}$ when this does not lead to confusion. We will also use the notation $a \prec b$ for $a \preceq b$ but $b \not\preceq a$, while $a \prec^1 b$ means that $a \prec b$ and there is no $c \in |\mathfrak{A}|$ such that $a \prec c \prec b$. An element $a \in |\mathfrak{A}|$ is *greatest* if $b \preceq a$ for all $b \in |\mathfrak{A}|$, and *maximal* if $b \not\preceq a$ for any $b \in |\mathfrak{A}|$. *Least* and *minimal* are defined analogously.

A partial order \mathfrak{A} is a *tree* if it has a greatest element r , and for every $x \in |\mathfrak{A}|$, $\uparrow x$ is finite and linearly ordered. Note that in our presentation, the leaves of a tree are its minimal elements.

2.2 Notions from topology

Let us recall the definition of topological spaces and continuous functions:

Definition 2.1. A topological space is a pair $\mathfrak{X} = (|\mathfrak{X}|, \mathcal{T}_{\mathfrak{X}})$, where $|\mathfrak{X}|$ is a set and $\mathcal{T}_{\mathfrak{X}}$ a family of subsets of $|\mathfrak{X}|$ satisfying

1. $\emptyset, |\mathfrak{X}| \in \mathcal{T}_{\mathfrak{X}}$;
2. if $U, V \in \mathcal{T}_{\mathfrak{X}}$ then $U \cap V \in \mathcal{T}_{\mathfrak{X}}$ and
3. if $\mathcal{O} \subseteq \mathcal{T}_{\mathfrak{X}}$ then $\bigcup \mathcal{O} \in \mathcal{T}_{\mathfrak{X}}$.

The elements of $\mathcal{T}_{\mathfrak{X}}$ are called open sets. Complements of open sets are closed sets. If $x \in U \in \mathcal{T}_{\mathfrak{X}}$, we say that U is a neighborhood of x .

If $\mathfrak{X}, \mathfrak{Y}$ are topological spaces, a function $f : |\mathfrak{X}| \rightarrow |\mathfrak{Y}|$ is continuous if $f^{-1}(V)$ is open whenever $V \subseteq |\mathfrak{Y}|$ is open, and open if $f(U)$ is open whenever $U \subseteq |\mathfrak{X}|$ is open.

Given a set $A \subseteq |\mathfrak{X}|$, its *interior*, denoted A° , is defined by

$$A^\circ = \bigcup \{U \in \mathcal{T}_{\mathfrak{X}} : U \subseteq A\}.$$

Dually, we define the closure \overline{A} as $|\mathfrak{X}| \setminus (|\mathfrak{X}| \setminus A)^\circ$; this is the smallest closed set containing A .

Definition 2.2. A collection \mathcal{B} of subsets of a set X is a basis if 1. $\bigcup_{B \in \mathcal{B}} B = X$, and 2. whenever $B_0, B_1 \in \mathcal{B}$ and $x \in B_0 \cap B_1$, there exists $B_2 \subseteq B_0 \cap B_1$ such that $x \in B_2$.

Any basis \mathcal{B} generates a topology \mathcal{T} on X defined by letting $U \subseteq X$ be open if and only if, for every $x \in U$, there is $B \in \mathcal{B}$ with $x \in B$.

Topological spaces can be seen as a generalization of posets. Consider the topology \mathcal{D}_{\preceq} on $|\mathfrak{W}|$ given by setting $U \subseteq |\mathfrak{W}|$ to be open if and only if, whenever $w \in U$, we have $\downarrow w \subseteq U$ (so that the sets of the form $\downarrow w$ provide a basis for \mathcal{D}_{\preceq}). We call \mathcal{D}_{\preceq} the *down-set topology* of \preceq . Similarly, the family of upwards-closed subsets of $|\mathfrak{W}|$ will be denoted by \mathcal{U}_{\preceq} , and is the *up-set topology* of \preceq . Topologies of this form are Aleksandroff topologies [1]:

Definition 2.3. A topological space \mathfrak{A} is an Aleksandroff space if any one of the following equivalent conditions occurs:

1. whenever $\mathcal{O} \subset \mathcal{T}_{\mathfrak{A}}$, then $\bigcap \mathcal{O} \in \mathcal{T}_{\mathfrak{A}}$;
2. every $x \in |\mathfrak{A}|$ has a least neighborhood;
3. there is a preorder \preceq on $|\mathfrak{A}|$ such that $\mathcal{T}_{\mathfrak{A}} = \mathcal{D}_{\preceq}$;
4. there is a preorder \preceq on $|\mathfrak{A}|$ such that $\mathcal{T}_{\mathfrak{A}} = \mathcal{U}_{\preceq}$.

It is readily verified that if $\mathcal{T}_{\mathfrak{A}} = \mathcal{D}_{\preceq}$ then \preceq is uniquely defined, and we will denote it by $\preceq_{\mathfrak{A}}$. We remark that intuitionistic logic cannot distinguish between preordered sets and partially ordered sets, so without loss of generality we may work only with Aleksandroff spaces generated by a poset. We will tacitly identify $(|\mathfrak{A}|, \preceq_{\mathfrak{A}})$ with $(|\mathfrak{A}|, \mathcal{T}_{\mathfrak{A}})$.

Observe that all finite topological spaces are Aleksandroff; in fact, all *locally* finite spaces are Aleksandroff. The following is easily verified:

Lemma 2.4. *Say that a topological space \mathfrak{A} is locally finite if every point has a neighborhood U such that $\#U < \omega$. Then, if \mathfrak{A} is locally finite, it follows that \mathfrak{A} is Aleksandroff.*

It is also useful to characterize the continuous functions on a poset:

Lemma 2.5. *If $\mathfrak{W}, \mathfrak{V}$ are preorders and $g: |\mathfrak{W}| \rightarrow |\mathfrak{V}|$, then g is continuous with respect to the down-set topologies on $\mathfrak{W}, \mathfrak{V}$ if and only if, whenever $v \preceq_{\mathfrak{W}} w$, it follows that $g(v) \preceq_{\mathfrak{V}} g(w)$.*

In other words, continuous maps on posets are simply non-decreasing maps.

3 Syntax and Semantics

In this section we will introduce the logic ITLC and its semantics. It will be convenient to first define an extended logic ITLC*, containing both our logic and Kremer's logic.

Fix a countably infinite set \mathbb{P} of propositional variables. The *full language* \mathcal{L}^* is defined by the grammar

$$\perp \mid p \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \rightarrow \psi \mid \circ \varphi \mid \diamond \varphi \mid \square \varphi \mid \exists \varphi \mid \forall \varphi,$$

where $p \in \mathbb{P}$. As usual, we use $\neg \varphi$ as a shorthand for $\varphi \rightarrow \perp$. We read \circ as ‘next’, \diamond as ‘eventually’, and \square as ‘henceforth’. We will denote the set of subformulas of $\varphi \in \mathcal{L}^*$ by $\text{sub}(\varphi)$. The length of φ will be denoted $\|\varphi\|$. We will also be interested in certain sublanguages of \mathcal{L}^* , which only allow some set of modalities $M \subseteq \{\circ, \diamond, \square, \exists, \forall\}$, and we will denote such a fragment by $\mathcal{L}^* \upharpoonright M$. Note that $\mathcal{L}^* \upharpoonright M$ always contains all Booleans, even though they will not be listed in M . We will be primarily interested in the language $\mathcal{L} = \mathcal{L}^* \upharpoonright \{\circ, \diamond, \forall\}$.

Formulas of \mathcal{L}^* are interpreted on dynamical systems over topological spaces, or *dynamic topological systems*.

Definition 3.1. A dynamical system is a triple $\mathfrak{X} = (|\mathfrak{X}|, \mathcal{T}_{\mathfrak{X}}, f_{\mathfrak{X}})$, where $(|\mathfrak{X}|, \mathcal{T}_{\mathfrak{X}})$ is a topological space and $f_{\mathfrak{X}} : |\mathfrak{X}| \rightarrow |\mathfrak{X}|$ is a continuous function.

A valuation on \mathfrak{X} is a function $\llbracket \cdot \rrbracket : \mathcal{L}^* \rightarrow \mathcal{T}_{\mathfrak{X}}$ such that

$$\begin{aligned} \llbracket \perp \rrbracket &= \emptyset \\ \llbracket \varphi \wedge \psi \rrbracket &= \llbracket \varphi \rrbracket \cap \llbracket \psi \rrbracket \\ \llbracket \varphi \vee \psi \rrbracket &= \llbracket \varphi \rrbracket \cup \llbracket \psi \rrbracket \\ \llbracket \varphi \rightarrow \psi \rrbracket &= ((|\mathfrak{X}| \setminus \llbracket \varphi \rrbracket) \cup \llbracket \psi \rrbracket)^\circ \\ \llbracket \circ \varphi \rrbracket &= f_{\mathfrak{X}}^{-1} \llbracket \varphi \rrbracket \\ \llbracket \Diamond \varphi \rrbracket &= \bigcup_{n < \omega} f_{\mathfrak{X}}^{-n} \llbracket \varphi \rrbracket \\ \llbracket \Box \varphi \rrbracket &= \left(\bigcap_{n < \omega} f_{\mathfrak{X}}^{-n} \llbracket \varphi \rrbracket \right)^\circ \\ \llbracket \exists \varphi \rrbracket &= |\mathfrak{X}| \text{ if } \llbracket \varphi \rrbracket \neq \emptyset \text{ and } \emptyset \text{ otherwise;} \\ \llbracket \forall \varphi \rrbracket &= |\mathfrak{X}| \text{ if } \llbracket \varphi \rrbracket = |\mathfrak{X}| \text{ and } \emptyset \text{ otherwise.} \end{aligned}$$

A dynamical system \mathfrak{X} equipped with a valuation $\llbracket \cdot \rrbracket_{\mathfrak{X}}$ is a dynamical model.

Note that, for a propositional variable p , $\llbracket p \rrbracket$ can be any subset of $|\mathfrak{X}|$, provided it is open. The rest of the clauses are standard from either intuitionistic or temporal logic, with the exception of $\Box \varphi$. The interpretation of the latter is due to Kremer, and we will discuss it further in the next section.

As a general convention, we let structures inherit the properties of its components, so that for example an Alexandroff dynamical model is a dynamical model \mathfrak{A} such that $(\mathfrak{A}, \mathcal{T}_{\mathfrak{A}})$ is an Aleksandroff space. In the setting of Aleksandroff models, the semantics for implication simplify somewhat.

Lemma 3.2. Let \mathfrak{A} be a dynamical model based on an Aleksandroff space, and φ be any formula. Then,

1. If $\mathcal{T}_{\mathfrak{A}} = \mathcal{D}_{\preccurlyeq}$, $x \in |\mathfrak{A}|$ and $\varphi, \psi \in \mathcal{L}^*$, then $x \in \llbracket \varphi \rightarrow \psi \rrbracket_{\mathfrak{A}}$ if and only if, for all $y \preccurlyeq x$, if $y \in \llbracket \varphi \rrbracket_{\mathfrak{A}}$, then $y \in \llbracket \psi \rrbracket_{\mathfrak{A}}$.
2. Similarly, if $\mathcal{T}_{\mathfrak{A}} = \mathcal{U}_{\succcurlyeq}$, $x \in |\mathfrak{A}|$ and $\varphi, \psi \in \mathcal{L}^*$, then $x \in \llbracket \varphi \rightarrow \psi \rrbracket_{\mathfrak{A}}$ if and only if, for all $y \succcurlyeq x$, if $y \in \llbracket \varphi \rrbracket_{\mathfrak{A}}$, then $y \in \llbracket \psi \rrbracket_{\mathfrak{A}}$.
3. $\llbracket \Box \varphi \rrbracket = \bigcap_{n < \omega} f^{-n} \llbracket \varphi \rrbracket$.

Proof. For the first claim, observe that if $x \in E \subseteq |\mathfrak{A}|$, then $x \in E^\circ$ if and only if $\downarrow x \subseteq E$. From this and the definition of $\llbracket \varphi \rightarrow \psi \rrbracket_{\mathfrak{A}}$, the claim follows. The second claim is analogous, but replacing \preccurlyeq by \succcurlyeq . For the third, since in an Aleksandroff space we have that infinite intersections of open sets are open, it follows that $\bigcap_{n < \omega} f^{-n} \llbracket \varphi \rrbracket$ is open, and thus

$$\llbracket \Box \varphi \rrbracket = \left(\bigcap_{n < \omega} f^{-n} \llbracket \varphi \rrbracket \right)^\circ = \bigcap_{n < \omega} f^{-n} \llbracket \varphi \rrbracket. \quad \square$$

Next, we define validity in the usual way:

Definition 3.3. Given a class \mathcal{C} of dynamical models, we say that $\varphi \in \mathcal{L}^*$ is valid over \mathcal{C} if, for every $\mathfrak{X} \in \mathcal{C}$, $\llbracket \varphi \rrbracket_{\mathfrak{X}} = |\mathfrak{X}|$.

We define the logics ITLC and ITLC* to be the set of formulas of \mathcal{L} and \mathcal{L}^* , respectively, that are valid over the class of all dynamical models.

Example 3.4. *The formula*

$$\varphi = \forall(p \vee \neg p) \rightarrow (\neg \Diamond p \rightarrow \Diamond p)$$

is valid on any dynamical model based on an Aleksandroff space. To see this, suppose that \mathfrak{A} is any such model with $\mathcal{T}_{\mathfrak{A}} = \mathcal{D}_{\preceq}$, and suppose that $w \in \llbracket \forall(p \vee \neg p) \rrbracket_{\mathfrak{A}}$. Suppose further that $w \in \llbracket \neg \Diamond p \rrbracket_{\mathfrak{A}}$. Then there is $v \preceq w$ such that $v \in \llbracket \Diamond p \rrbracket_{\mathfrak{A}}$, from which we obtain $f_{\mathfrak{A}}^n(v) \in \llbracket p \rrbracket_{\mathfrak{A}}$ for some n .

But, we must have that $f_{\mathfrak{A}}^n(w) \in \llbracket p \rrbracket_{\mathfrak{A}}$ as well; for otherwise, since $f_{\mathfrak{A}}^n(v) \preceq f_{\mathfrak{A}}^n(w)$, we would have that $f_{\mathfrak{A}}^n(w) \notin \llbracket p \vee \neg p \rrbracket_{\mathfrak{A}}$, and hence $w \notin \llbracket \forall(p \vee \neg p) \rrbracket_{\mathfrak{A}}$, a contradiction. It follows that $\llbracket \varphi \rrbracket_{\mathfrak{A}} = |\mathfrak{A}|$, and thus φ is valid over the class of Aleksandroff dynamical systems.

However, as we will see later, the above formula φ is not valid over the class of all dynamical models. We remark that our full language can be simplified somewhat; in particular, it admits \exists -elimination.

Lemma 3.5. *Given any dynamic model \mathfrak{X} and any formula φ , $\llbracket \exists \varphi \rrbracket_{\mathfrak{X}} = \llbracket \neg \forall \neg \varphi \rrbracket_{\mathfrak{X}}$.*

Proof. Note that $\llbracket \exists \varphi \rrbracket_{\mathfrak{X}} \in \{\emptyset, |\mathfrak{X}|\}$. If $\llbracket \exists \varphi \rrbracket_{\mathfrak{X}} = \emptyset$, then $\llbracket \varphi \rrbracket_{\mathfrak{X}} = \emptyset$, and thus $\llbracket \neg \varphi \rrbracket_{\mathfrak{X}} = |\mathfrak{X}|$. It follows that $\llbracket \forall \neg \varphi \rrbracket_{\mathfrak{X}} = |\mathfrak{X}|$, and thus $\llbracket \neg \forall \neg \varphi \rrbracket_{\mathfrak{X}} = \emptyset$.

Otherwise, $\llbracket \exists \varphi \rrbracket_{\mathfrak{X}} = |\mathfrak{X}|$, which means that $\llbracket \varphi \rrbracket_{\mathfrak{X}} \neq \emptyset$. Hence $\llbracket \neg \varphi \rrbracket_{\mathfrak{X}} \neq |\mathfrak{X}|$; but then, $\llbracket \forall \neg \varphi \rrbracket_{\mathfrak{X}} = \emptyset$, and thus $\llbracket \neg \forall \neg \varphi \rrbracket_{\mathfrak{X}} = |\mathfrak{X}|$. \square

Thus we may turn our attention to $\mathcal{L}^* \upharpoonright \{\circ, \Diamond, \Box, \forall\}$, or, generally speaking, to languages without \exists . Observe, however, that $\neg \exists \neg \varphi$ is not always equivalent to $\forall \varphi$, nor can we define \Diamond, \Box in terms of each other using the classical definitions.

Example 3.6. *Consider the dynamical model \mathfrak{X} where $(|\mathfrak{X}|, \mathcal{T}_{\mathfrak{X}})$ is the real line \mathbb{R} with its usual topology, and f is the constant function $f(x) \equiv 0$. Suppose that $\llbracket p \rrbracket = \mathbb{R} \setminus 0$. Then, $\llbracket \neg p \rrbracket = \emptyset$, since $\{0\}$ has empty interior. It follows that $\llbracket \neg \exists \neg p \rrbracket = \mathbb{R}$. On the other hand, $\llbracket p \rrbracket \neq \mathbb{R}$, hence $\llbracket \forall p \rrbracket \neq \mathbb{R}$, and $\neg \exists \neg p \leftrightarrow \forall p$ is not valid.*

By similar reasoning, $\llbracket \Diamond \neg p \rrbracket = \emptyset$, hence $\llbracket \neg \Diamond \neg p \rrbracket = \mathbb{R}$, yet $\llbracket \Box p \rrbracket = \emptyset$ since, for arbitrary $x \in \mathbb{R}$, $f(x) = 0 \notin \llbracket p \rrbracket$. It follows that $\neg \Diamond \neg p \leftrightarrow \Box p$ is not valid.

Finally, $\llbracket \Box \neg p \rrbracket = \emptyset$, hence $\llbracket \neg \Box \neg p \rrbracket = \mathbb{R}$. However, $0 \notin \llbracket \Diamond p \rrbracket$, so that once again, $\neg \Box \neg p \leftrightarrow \Diamond p$ is not valid.

Because of this, we have included \forall in \mathcal{L} but not \exists ; since the latter is definable, we lose nothing by omitting it. On the other hand, \Box cannot be defined from \Diamond in the obvious way, but we omit it for technical reasons that will be discussed later.

4 Related Logics

Before we continue our analysis of ITLC, let us mention some related systems. First we observe that ITLC^* is an extension of the logic introduced by Kremer in [15]. To be precise, Kremer considers the language $\mathcal{L}^K = \mathcal{L}^* \upharpoonright \{\circ, \Box\}$. There

he also presents the next example, which will help elucidate the semantics of $\llbracket \Box \varphi \rrbracket$.

Example 4.1. Consider the dynamical system \mathfrak{X} where $(|\mathfrak{X}|, \mathcal{T}_{\mathfrak{X}})$ is \mathbb{R} with its usual topology, and $f = f_{\mathfrak{X}}: |\mathfrak{X}| \rightarrow |\mathfrak{X}|$ is given by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 2x & \text{otherwise.} \end{cases}$$

Let $\llbracket p \rrbracket = (-\infty, 1)$ (the valuations of other atoms do not matter).

Observe that, for all n , $f^n(0) = 0 \in \llbracket p \rrbracket$. Thus $0 \in \bigcap_{n < \omega} f^{-n} \llbracket \varphi \rrbracket$. Moreover, if $x < 0$, then $x \in \llbracket p \rrbracket$ and $f(x) = 0$, so once again $x \in \bigcap_{n < \omega} f^{-n} \llbracket \varphi \rrbracket$.

On the other hand, if $x > 0$, then $f^n(x) = 2^n x$. Since $2^n \rightarrow \infty$ as $n \rightarrow \infty$, we have that, for n large enough, $2^n x > 1$, and thus $f^n(x) \notin \llbracket p \rrbracket$. It follows that $x \notin \bigcap_{n < \omega} f^{-n} \llbracket \varphi \rrbracket$.

But then we have that

$$\bigcap_{n < \omega} f^{-n} \llbracket \varphi \rrbracket = (-\infty, 0];$$

observe that this set is not open. Since intuitionistic truth values must always be open, we need to “approximate” it by an open set; this is done in the standard way by taking the interior, and thus

$$\llbracket \Box p \rrbracket = \left(\bigcap_{n < \omega} f^{-n} \llbracket \varphi \rrbracket \right)^{\circ} = (-\infty, 0).$$

This example shows that some care must be taken when working with \Box , and as we will discuss in Example 5.4, the decidability proof we will present would not go through in the presence of \Box (at least not without some non-trivial modifications). Note, however, that in view of Lemma 3.2, this issue disappears when working over the class of Aleksandroff spaces.

Aleksandroff spaces will allow us to compare our semantics with the intuitionistic functional frames presented in [12] (the semantics used in [11] are similar). Kojima and Igarashi use relational structures (W, \preceq, f) , where $f \circ \preceq = \preceq \circ f$. They let $w \in W$ satisfy $\varphi \rightarrow \psi$ if, whenever $v \succ w$, if v satisfies φ , it also satisfies ψ ; note that these are precisely the truth conditions with respect to the up-set topology \mathcal{U}_{\preceq} . The reader may readily verify that the condition $\preceq \circ f \supseteq f \circ \preceq$ is equivalent to monotonicity, or continuity, of f . On the other hand, $\preceq \circ f \subseteq f \circ \preceq$ tells us that $\uparrow f(w) \subseteq f[\uparrow w]$; in other words, that f is an open map.

Hence, intuitionistic functional models are precisely the Aleksandroff dynamical systems with an *interior* (i.e., continuous and open) map. Let us denote this class of systems by \mathbf{AI} , and hence let \mathbf{ITLAI} be the set of formulas of \mathcal{L} valid in \mathbf{AI} .³ It is immediate that $\mathbf{ITLC} \subseteq \mathbf{ITLAI}$; as we will see, the inclusion is in fact strict.

³We remark that [12] considers instead a weaker logic over $\mathcal{L}^* \upharpoonright \{\circ\}$, which is sound, but not complete, for the class of intuitionistic functional models.

Finally, it will be convenient to compare ITLC to dynamic topological logic. Since the base logic is classical, we may use a simpler syntax for DTL, using the language \mathcal{L}^\blacksquare given by the grammar

$$\perp \mid p \mid \varphi \rightarrow \psi \mid \blacksquare\varphi \mid \circ\varphi \mid \Box\varphi \mid \forall\varphi.$$

We can then define $\neg, \wedge, \vee, \blacklozenge, \diamond$ using standard classical validities.

Given a dynamical system, \mathfrak{X} , a *classical valuation on \mathfrak{X}* is a function $\llbracket \cdot \rrbracket^C : \mathcal{L}^\blacksquare \rightarrow \mathcal{T}_\mathfrak{X}$ such that

$$\begin{aligned} \llbracket \perp \rrbracket^C &= \emptyset \\ \llbracket \varphi \rightarrow \psi \rrbracket^C &= (|\mathfrak{X}| \setminus \llbracket \varphi \rrbracket^C) \cup \llbracket \psi \rrbracket^C \\ \llbracket \blacksquare\varphi \rrbracket^C &= (\llbracket \varphi \rrbracket^C)^\circ \\ \llbracket \circ\varphi \rrbracket^C &= f_\mathfrak{X}^{-1} \llbracket \varphi \rrbracket^C \\ \llbracket \Box\varphi \rrbracket^C &= \bigcap_{n < \omega} f_\mathfrak{X}^{-n} \llbracket \varphi \rrbracket^C \\ \llbracket \forall\varphi \rrbracket^C &= |\mathfrak{X}| \text{ if } \llbracket \varphi \rrbracket^C = |\mathfrak{X}| \text{ and } \emptyset \text{ otherwise.} \end{aligned}$$

Our intuitionistic temporal logic may readily be interpreted in DTL, via the Gödel-Tarski translation \cdot^\blacksquare .

Definition 4.2. Given $\varphi \in \mathcal{L}^*$, we define $\varphi^\blacksquare \in \mathcal{L}^\blacksquare$ by recursively replacing each subformula ψ of φ by $\blacksquare\psi^\blacksquare$.

For example, $(\Box p \rightarrow \perp)^\blacksquare = \blacksquare(\blacksquare\Box\blacksquare p \rightarrow \blacksquare\perp)$. Then, the following can be verified by a simple induction on φ :

Lemma 4.3. Let $\varphi \in \mathcal{L}^*$, and \mathfrak{X} be any dynamic topological system. Suppose that $\llbracket \cdot \rrbracket$ is an intuitionistic valuation and $\llbracket \cdot \rrbracket^C$ a classical valuation such that, for every atom p , $(\llbracket p \rrbracket^C)^\circ = \llbracket p \rrbracket$. Then, for every formula φ , $\llbracket \varphi \rrbracket = \llbracket \varphi^\blacksquare \rrbracket^C$.

In fact, the translation can be simplified somewhat; it suffices to place \blacksquare in front of atoms and formulas of the forms $\varphi \rightarrow \psi$ or $\Box\varphi$. From 4.3 and the fact that DTL is computably enumerable [8], we immediately obtain the following.

Theorem 4.4. The logic ITLC* is computably enumerable.

However, DTL is undecidable [13], and hence Theorem 4.4 does not settle the decidability of ITLC*. We do not know whether full ITLC* is decidable; however, in the rest of this article we will show that this is indeed the case for ITLC.

5 Labeled systems

Our decidability proof is based on non-deterministic quasimodels, introduced in [8]. In this section we will introduce labeled systems, which generalize both quasimodels and dynamical models, thus allowing us to view both of them in a unified framework. Let us write $\Sigma \in \mathcal{L}$ to indicate that $\Sigma \subseteq \mathcal{L}$ is finite and is

closed under subformulas. Given $\Sigma \in \mathcal{L}$, we say that a set of formulas $\Phi \subseteq \Sigma$ is a Σ -type if:

1. $\perp \notin \Sigma$
2. if $\varphi \wedge \psi \in \Sigma$, then $\varphi \wedge \psi \in \Phi$ if and only if $\varphi, \psi \in \Phi$,
3. if $\varphi \vee \psi \in \Sigma$, then $\varphi \vee \psi \in \Phi$ if and only if $\varphi \in \Phi$ or $\psi \in \Phi$,
4. if $\varphi \rightarrow \psi \in \Sigma$, then $\varphi \rightarrow \psi \in \Phi$ implies that $\varphi \notin \Phi$ or $\psi \in \Phi$, and
5. if $\Diamond\psi \in \Sigma$ and $\psi \in \Phi$, then $\Diamond\psi \in \Phi$.

The set of Σ -types will be denoted by \mathbb{T}_Σ . Note that \mathbb{T}_Σ is partially ordered by \subseteq , and we will endow it with the up-set topology \mathcal{U}_\subseteq . For $\Phi \in \mathbb{T}_\Sigma$, say that a formula $\varphi \rightarrow \psi \in \Sigma$ is a *defect* of Φ if $\varphi \rightarrow \psi \notin \Phi$ but $\varphi \notin \Phi$. The set of defects of Φ will be denoted $\partial_\Sigma \Phi$, or simply $\partial\Phi$ when Σ is clear from context.

Definition 5.1 (labeled space). *Fix $\Sigma \in \mathcal{L}$. A Σ -labeled space is a triple $\mathfrak{W} = (|\mathfrak{W}|, \mathcal{T}_\mathfrak{W}, \ell_\mathfrak{W})$, where $(|\mathfrak{W}|, \mathcal{T}_\mathfrak{W})$ is a topological space and $\ell_\mathfrak{W}: |\mathfrak{W}| \rightarrow \mathbb{T}_\Sigma$ is a continuous function such that for all $w \in |\mathfrak{W}|$, whenever $\varphi \rightarrow \psi \in \partial\ell_\mathfrak{W}(w)$ and U is any neighborhood of w , there is $v \in U$ such that $\varphi \in \ell_\mathfrak{W}(v)$ and $\psi \notin \ell_\mathfrak{W}(v)$. Such a v revokes $\varphi \rightarrow \psi$.*

We say that \mathfrak{W} falsifies $\varphi \in \mathcal{L}$ if $\varphi \in \Sigma \setminus \ell_\mathfrak{W}(w)$ for some $w \in |\mathfrak{W}|$. We say that $\ell_\mathfrak{W}$ is extensional if, for every $w \in |\mathfrak{W}|$ and every $\forall\varphi \in \Sigma$, we have that $\forall\varphi \in \ell_\mathfrak{W}(w)$ if and only if $\varphi \in \ell_\mathfrak{W}(v)$ for every $v \in |\mathfrak{W}|$.

As usual, we may write ℓ instead of $\ell_\mathfrak{W}$ when this does not lead to confusion. Since we have endowed \mathbb{T}_Σ with the topology \mathcal{U}_\subseteq , the continuity of ℓ means that for every $w \in |\mathfrak{W}|$, there is a neighborhood U of w such that, whenever $v \in U$, $\ell_\mathfrak{W}(w) \subseteq \ell_\mathfrak{W}(v)$.

Note that not every subset U of $|\mathfrak{W}|$ gives rise to a substructure that is also a labeled space; however, this is the case when U is open.

Lemma 5.2. *If $\Sigma \in \mathcal{L}$, \mathfrak{W} is a Σ -labeled space, and $U \subseteq |\mathfrak{W}|$ is open, then $\mathfrak{W} \upharpoonright U$ is a Σ -labeled space.*

For our purposes, a *continuous relation* on a topological space is a relation under which the preimage of any open set is open (note that this is not the standard definition of a continuous relation, which is more involved).

Definition 5.3 (labeled systems and sensible relations). *Let $\Sigma \in \mathcal{L}$. Suppose that $\Phi, \Psi \in \mathbb{T}_\Sigma$. The ordered pair (Φ, Ψ) is sensible if*

1. *for all $\circ\varphi \in \Sigma$, $\circ\varphi \in \Phi$ if and only if $\varphi \in \Psi$, and*
2. *for all $\Diamond\varphi \in \Sigma$, $\Diamond\varphi \in \Phi$ if and only if $\varphi \in \Phi$ or $\Diamond\varphi \in \Psi$.*

Likewise, a pair (w, v) of worlds in a labeled space \mathfrak{W} is sensible if $(\ell_{\mathfrak{W}}(w), \ell_{\mathfrak{W}}(v))$ is sensible.

A continuous relation $S \subseteq |\mathfrak{W}| \times |\mathfrak{W}|$ is sensible if every pair in S is sensible.

Further, S is ω -sensible if it is serial and whenever $\Diamond\varphi \in \ell(w)$, there are $n \geq 0$ and v such that $w S^n v$ and $\varphi \in \ell(v)$.

A labeled system is a labeled space \mathfrak{W} equipped with a sensible relation $S_{\mathfrak{W}} \subseteq |\mathfrak{W}| \times |\mathfrak{W}|$; if moreover $\ell_{\mathfrak{W}}$ is extensional and $S_{\mathfrak{W}}$ is ω -sensible, we say that \mathfrak{W} is a well Σ -labeled system.

Well-labeled systems generalize dynamical models in the following way. If \mathfrak{X} is a dynamical model and $x \in |\mathfrak{X}|$, assign a Σ -type $\ell_{\mathfrak{X}}(x)$ to x given by

$$\ell_{\mathfrak{X}}(x) = \{\psi \in \Sigma : x \in \llbracket \psi \rrbracket_{\mathfrak{X}}\}.$$

We may also set $S_{\mathfrak{X}} = f_{\mathfrak{X}}$; it is obvious that $\ell_{\mathfrak{X}}$ is extensive and $S_{\mathfrak{X}}$ is ω -sensible.

Example 5.4. Example 4.1 can be used to show that there are some difficulties when treating \Box using labeled systems. In that example, we have that $\llbracket \Box p \rrbracket = (-\infty, 0)$. Thus, for instance, $-1 \in \llbracket \Box p \rrbracket$, but $f(-1) = 0 \notin \llbracket \Box p \rrbracket$. If we wanted to label our model using $\Sigma = \{p, \Box p\}$, we would have to set $\ell(-1) = \{p, \Box p\}$ and $\ell(0) = \{p\}$. Nevertheless, by the semantics of $\Box p$, we know that $f^n(0) \in \llbracket p \rrbracket$ for all $n < \omega$, yet this information is not recorded in $\ell(0)$.

We could get around this issue by using classical semantics and the Gödel-Tarski translation, where $(\Box p)^{\blacksquare} = \blacksquare \Box \blacksquare p$. If we use $\Sigma = \text{sub}(\blacksquare \Box \blacksquare p)$, we would have $\blacksquare \Box \blacksquare p \in \ell(-1)$, but also $\Box \blacksquare p \in \ell(-1)$ and hence $\Box \blacksquare p \in \ell(0)$. With this, 0 would ‘remember’ that all of its temporal successors must satisfy $\blacksquare p$. However, note that $\Box \blacksquare p \notin \ell(x)$ for any $x > 0$, and hence ℓ is no longer continuous. This is a problem for us, since the continuity of ℓ is used in an essential way (for example, in Lemma 9.7) to bound the size of our quasimodels.

Another useful class of labeled systems is given by quasimodels:

Definition 5.5 (quasimodel). A weak Σ -quasimodel is a Σ -labeled system \mathfrak{Q} such that $\mathcal{T}_{\mathfrak{Q}}$ is locally finite, and equal to the down-set topology for a partial order $\preceq_{\mathfrak{Q}}$. If moreover \mathfrak{Q} is a well Σ -labeled system, then we say that \mathfrak{Q} is a Σ -quasimodel.

Example 5.6. Recall from Example 3.4 that the formula

$$\varphi = \forall(p \vee \neg p) \rightarrow (\neg \neg \Diamond p \rightarrow \Diamond p)$$

is valid on any Aleksandroff dynamical model. However, Figure 1 exhibits a

quasimodel⁴ falsifying φ . Let $\psi = \neg\neg\Diamond p \rightarrow \Diamond p$, and define ℓ to be

$$\begin{aligned}\ell(u) &= \{\forall(\mathbf{p} \vee \neg\mathbf{p}), (\mathbf{p} \vee \neg\mathbf{p}), \neg\neg\Diamond\mathbf{p}, \neg\mathbf{p}\} \\ \ell(v) &= \{\Diamond\mathbf{p}, \neg\mathbf{p}, (\mathbf{p} \vee \neg\mathbf{p}), \varphi, \psi, \forall(\mathbf{p} \vee \neg\mathbf{p}), \neg\neg\Diamond\mathbf{p}\} \\ \ell(w) &= \{\mathbf{p}, (\mathbf{p} \vee \neg\mathbf{p}), \varphi, \psi, \forall(\mathbf{p} \vee \neg\mathbf{p}), \neg\neg\Diamond\mathbf{p}, \Diamond\mathbf{p}\}.\end{aligned}$$

We include the full labels for completeness, but the most relevant formulas are displayed in **boldface**. In particular, observe that p belongs only to $\ell(w)$, and $\varphi \notin \ell(u)$; the latter means that u falsifies φ , and thus φ is not valid over the class of $\text{sub}(\varphi)$ -quasimodels. As we will see, this implies that φ is not valid over the class of dynamical models.

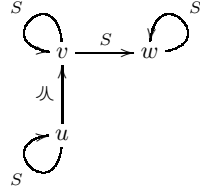


Figure 1: A Σ -quasimodel with $\Sigma = \text{sub}(\varphi)$.

6 Limit Models

Note that, if \mathfrak{Q} is a quasimodel, $S_{\mathfrak{Q}}$ is not necessarily a function, and thus we may not view \mathfrak{Q} directly as a dynamical model. However, we can extract a dynamical model $\vec{\mathfrak{Q}}$ from it via an unwinding construction. We will call $\vec{\mathfrak{Q}}$ the limit model of \mathfrak{Q} . The points of $\vec{\mathfrak{Q}}$ are realizing sequences, which we define next.

6.1 Realizing sequences

A *path* in a quasimodel \mathfrak{Q} is any finite sequence $(w_n)_{n < \xi}$ with $\xi \leq \omega$ such that $w_n S w_{n+1}$ whenever $n+1 < \xi$. An infinite path $\vec{w} = (w_n)_{n < \omega}$ is *realizing* if for all $n \geq 0$ and $\Diamond\psi \in \ell(w_n)$ there exists $k \geq n$ such that $\psi \in \ell(w_k)$.

Denote the set of realizing paths by $|\vec{\mathfrak{Q}}|$. This will be the domain of the limit model of \mathfrak{Q} . The transition function on $|\vec{\mathfrak{Q}}|$ will be the ‘shift’ operator, defined by $\sigma((w_n)_{n < \omega}) = (w_{n+1})_{n < \omega}$. This simply removes the first element in the sequence.

⁴Compare to [8], where we use a very similar quasimodel to falsify the formula $\psi = \Box\blacksquare p \rightarrow \blacksquare\Box p$, also valid over Aleksandrov systems. Note, however, that this ψ uses the classical semantics and thus we use a slightly more complicated example here.

For our construction to work we must guarantee that there are ‘enough’ realizing paths, in the sense of the following definition.

Definition 6.1 (extensive set). *Let $\Sigma \in \mathcal{L}$, \mathfrak{Q} be a Σ -typed non-deterministic quasimodel, and $E \subseteq |\vec{\mathfrak{Q}}|$.*

Then, E is extensive if 1. E is closed under σ , and 2. any finite path (w_0, w_1, \dots, w_N) in \mathfrak{Q} can be extended to an infinite (realizing) path $\vec{w} = (w_n)_{n < \omega} \in E$.

Lemma 6.2. *Let Σ be a finite set of formulas closed under subformulas. If \mathfrak{Q} is a Σ -quasimodel, then $|\vec{\mathfrak{Q}}|$ is extensive.*

Proof. It is obvious that $|\vec{\mathfrak{Q}}|$ is closed under σ .

Let (w_0, \dots, w_N) be a finite path and ψ_0, \dots, ψ_I be all formulas such that $\Diamond \psi_i \in \ell(w_N)$. Suppose inductively that we have constructed a path

$$(w_0, \dots, w_N, w_{N+1} \dots w_{N+K_i})$$

such that, for all $j < i$, $\psi_j \in \ell(w_n)$ for some $n \in [N, N + K_i]$. Because $S = S_{\mathfrak{Q}}$ is ω -sensible, we know that there exist k_i and $v_i \in S^{k_i}(w_{N+K_i})$ such that $\psi_i \in \ell(v_i)$. We can then define $w_{N+K_i}, \dots, w_{N+K_i+k_j} = v_i$ in such a way that $w_n S w_{n+1}$ for all $n < N + K_i + k_i$, and set $K_{i+1} = K_i + k_i$. Repeating this process, we obtain a path

$$(w_0, \dots, w_N, w_{N+1} \dots w_{N+K_I})$$

realizing all $\Diamond \psi_i \in \ell(w_N)$. Note that we may have $K_I = 0$ (if we never needed to add new worlds), in which case we choose arbitrary w_{N+1} such that $w_N S w_{N+1}$, using the seriality of S . We then repeat the process beginning with (w_0, \dots, w_{N+K_I}) , and continue countably many times until an infinite realizing path is formed. \square

Lemma 6.3. *Let \mathfrak{Q} be a Σ -quasimodel, $(w_n)_{n \leq N}$ a finite path, and v_0 be such that $v_0 \preccurlyeq w_0$. Then, there exists a path $(v_n)_{n \leq N}$ such that, for $n \leq N$, $v_n \preccurlyeq w_n$.*

Proof. This follows from the continuity of $S_{\mathfrak{Q}}$ by an easy induction on N . \square

6.2 Limit models

If $\Sigma \in \mathcal{L}$ and \mathfrak{Q} is a Σ -quasimodel, the relation \preccurlyeq induces a topology on W , as we have seen before, by letting open sets be those which are downwards closed under \preccurlyeq . Likewise, \preccurlyeq induces a very different topology on $|\vec{\mathfrak{Q}}|$, in a rather natural way:

Lemma 6.4. *For each $\vec{w} \in \vec{W}$ and $N \geq 0$ define*

$$\downarrow_n \vec{w} = \left\{ \{v_i\}_{i < \omega} \in |\vec{\mathfrak{Q}}| : \forall i \leq N, v_i \preccurlyeq w_i \right\}.$$

Then, the set $\mathcal{B}^{\preccurlyeq} = \left\{ \downarrow_n \vec{w} : \vec{w} \in \vec{W}, N \geq 0 \right\}$ forms a topological basis on \vec{W} .

Proof. To check that it satisfies 2.2.1, note that it is obvious that, given any path $\vec{w} \in |\vec{\Omega}|$, there is a basic set containing it (say, $\downarrow_0 \vec{w}$). Hence $|\vec{\Omega}| = \bigcup_{\vec{w} \in |\vec{\Omega}|} \downarrow_0 \vec{w}$. As for 2.2.2, assuming that $\vec{v} \in \downarrow_{n_0} \vec{w}_0 \cap \downarrow_{n_1} \vec{w}_1$, we observe that $\downarrow_{\max(n_0, n_1)} \vec{v} \subseteq \downarrow_{n_0} \vec{w}_0 \cap \downarrow_{n_1} \vec{w}_1$. \square

Definition 6.5. The topology $\mathcal{T}_{\vec{\Omega}}$ on $|\vec{\Omega}|$ is the topology generated by the basis \mathcal{B}^{\prec} .

Now that we have equipped Ω with a topology, we need a continuous transition function on it to have a dynamical system.

Lemma 6.6. The ‘shift’ operator $\sigma : |\vec{\Omega}| \rightarrow |\vec{\Omega}|$ is continuous under the topology $\mathcal{T}_{\vec{\Omega}}$.

Proof. Let $\vec{w} = (w_i)_{i < \omega}$ be a realizing path and $\downarrow_n \sigma(\vec{w})$ be a neighborhood of $\sigma(\vec{w})$. Then, if $\vec{v} \in \downarrow_{n+1} \vec{w}$, $w_i \preceq v_i$ for all $i \leq n+1$, so $w_{i+1} \preceq v_{i+1}$ for all $i \leq n$ and $\sigma(\vec{v}) \in \downarrow_n \sigma(\vec{w})$. Hence $\sigma(\downarrow_{n+1} \vec{w}) \subseteq \downarrow_n \sigma(\vec{w})$, and σ is continuous. \square

Finally, we will use ℓ to define a truth valuation: if p is a propositional variable, set $\llbracket p \rrbracket^\ell = \{ \vec{w} \in \vec{W} \in \ell(w_0) \}$.

We are now ready to assign a dynamic topological model to each quasimodel:

Definition 6.7 (limit model). Given a Σ -quasimodel Ω , define

$$\vec{\Omega} = (|\vec{\Omega}|, \mathcal{T}_{\vec{\Omega}}, \sigma, \llbracket \cdot \rrbracket^\ell)$$

to be the limit model of Ω .

Of course this model is only useful if $\llbracket \cdot \rrbracket^\ell$ corresponds with ℓ on all formulas of Σ , not just propositional variables. Fortunately, this turns out to be the case.

Lemma 6.8. Let $\Sigma \subseteq \mathcal{L}$, Ω be a Σ -quasimodel, $\vec{w} = (w_n)_{n \geq 0} \in |\vec{\Omega}|$ and $\varphi \in \Sigma$. Then,

$$\llbracket \varphi \rrbracket^\ell = \{ \vec{w} : \varphi \in \ell(w_0) \}.$$

Proof. The proof goes by standard induction of formulas. The induction steps for \wedge, \vee are immediate; here we will only treat the cases for $\rightarrow, \circ, \Diamond, \Box$.

Case 1: $\varphi = \psi \rightarrow \theta$ If $\varphi \in \ell(w_0)$, take the neighborhood $\downarrow_0 w_0$ of \vec{w} and consider $\vec{v} \in \downarrow_0 w_0$, so that $v_0 \preceq w_0$. Then, since Ω is a Σ -labeled space, it follows that, if $\psi \in \ell(v_0)$, then $\theta \in \ell(v_0)$; by the induction hypothesis, this means that if $\vec{v} \in \llbracket \psi \rrbracket^\ell$, then $\vec{v} \in \llbracket \theta \rrbracket^\ell$. It follows that $\vec{w} \in \llbracket \psi \rightarrow \theta \rrbracket^\ell$.

On the other hand, if $\psi \rightarrow \theta \notin \ell(w_0)$, there is $v_0 \succ w_0$ such that $\psi \in \ell(v_0)$ but $\theta \notin \ell(v_0)$. Consider any basic neighborhood $\downarrow_n \vec{w}$ of \vec{w} . Then, by Lemma 6.3, there exists a path $(v_0, \dots, v_n) \subseteq W$ such that, for all $i \leq n$, $w_i \preceq v_i$ and for $i < n$, $v_i \succ v_{i+1}$. Because $|\vec{\Omega}|$ is extensive, $\{v_i\}_{0 \leq i \leq n}$ can be extended to a realizing path $\vec{v} \in Y$. Then, $\vec{v} \in \downarrow_n \vec{w}$, and by induction hypothesis we have that $\vec{v} \in \llbracket \varphi \rrbracket^\ell \setminus \llbracket \theta \rrbracket^\ell$. Since n was arbitrary, we conclude that $\vec{w} \notin \llbracket \psi \rightarrow \theta \rrbracket^\ell$.

Case 2: $\varphi = \circ\psi$ This case follows from the fact that (w_0, w_1) is sensible and the induction hypothesis.

Case 3: $\varphi = \Diamond\psi$ Because \vec{w} is a realizing path, we have that if $\Diamond\psi \in \ell(w_0)$, $\psi \in \ell(w_n)$ for some $n \geq 0$. We can use the induction hypothesis to conclude that $\sigma^n(\vec{w}) \in \llbracket \psi \rrbracket^\ell$ and so $\vec{w} \in \llbracket \Diamond\psi \rrbracket^\ell$.

For the other direction, assume that $\Diamond\psi \notin \ell(w_0)$. For all n , (w_n, w_{n+1}) is sensible so an easy induction shows that $\psi \notin \ell(w_n)$ and $\sigma^n(\vec{w}) \notin \llbracket \psi \rrbracket^\ell$; since n was arbitrary, $\vec{w} \notin \llbracket \Diamond\psi \rrbracket^\ell$.

Case 4: $\varphi = \forall\psi$ If $\forall\psi \in \ell(w_0)$, then since ℓ is extensive, we can use the induction hypothesis to conclude that $\vec{v} \in \llbracket \psi \rrbracket^\ell$ for all $\vec{v} \in |\vec{\Omega}|$ and thus $\vec{w} \in \llbracket \forall\psi \rrbracket^\ell$. If $\forall\psi \notin \ell(w_0)$, there is $v_0 \in |\Omega|$ such that $\psi \notin \ell(v_0)$. We can extend v_0 to a realizing path \vec{v} and, by the induction hypothesis, $\vec{v} \notin \llbracket \psi \rrbracket^\ell$, hence $\vec{w} \notin \llbracket \forall\psi \rrbracket^\ell$. \square

We are now ready to prove the main theorem of this section, which in particular implies that ITLC is sound for the class of quasimodels.

Theorem 6.9. *Let $\Sigma \in \mathcal{L}$, and suppose that $\varphi \in \Sigma$ is falsified in a Σ -quasimodel Ω . Then, there exists $\vec{w}^* \in |\vec{\Omega}|$ such that $\vec{w}^* \notin \llbracket \varphi \rrbracket^\ell$.*

Proof. Pick $w^* \in |\Omega|$ such that $\varphi \notin \ell(w^*)$. By Lemma 6.2, w^* can be extended to a realizing path \vec{w}^* . It follows from Lemma 6.8 that $\vec{w}^* \notin \llbracket \varphi \rrbracket^\ell$. \square

Example 6.10. Recall from Examples 3.4 and 5.6 that the formula

$$\varphi = \forall(p \vee \neg p) \rightarrow (\neg\neg\Diamond p \rightarrow \Diamond p)$$

is valid over the class of Aleksandroff systems, but is falsifiable in a finite quasimodel. Thus it follows from Theorem 6.9 that φ is also falsifiable in a dynamical model.

It is instructive to attempt to construct such a model directly, rather than appealing to the theorem. To this end, let us refute φ on \mathbb{Q} . We will use f given by $f(x) = x + 1$. Define a set

$$D = \bigcup_{n < \omega} \mathbb{Q} \cap \left(n - \frac{1}{n+\pi}, n + \frac{1}{n+\pi} \right)$$

and let $\llbracket p \rrbracket = \mathbb{Q} \setminus D$. It is readily verified that $\frac{1}{n+\pi} \notin \mathbb{Q}$ for any $n \in \mathbb{N}$, and hence

$$\mathbb{Q} \cap \left(n - \frac{1}{n+\pi}, n + \frac{1}{n+\pi} \right) = \mathbb{Q} \cap \left[n - \frac{1}{n+\pi}, n + \frac{1}{n+\pi} \right],$$

so that D is both closed and open. It follows from this that $\llbracket \neg p \rrbracket = D$, and hence $\llbracket p \vee \neg p \rrbracket = \mathbb{Q}$, so that $\llbracket \forall(p \vee \neg p) \rrbracket = \mathbb{Q}$. In particular, $0 \in \llbracket \forall(p \vee \neg p) \rrbracket$.

Moreover, one can check that (a) $0 \notin \llbracket \Diamond p \rrbracket$, but (b) if $x \notin \mathbb{N}$, then $x \in \llbracket \Diamond p \rrbracket$. But, it readily follows from (b) that $0 \in \llbracket \neg\neg\Diamond p \rrbracket$, and hence $0 \notin \llbracket \varphi \rrbracket$.

In view of Example 3.4, it follows that $\text{ITLC} \subsetneq \text{ITLAI}$. Note that the disconnectedness of \mathbb{Q} is used in an essential way in this example; it is well-known that

$$\forall(p \vee \neg p) \rightarrow (\forall p \vee \forall \neg p)$$

is valid over the class of connected spaces [21], from which the validity of φ for connected systems is immediate.

7 Moments

We have seen that any valid $\varphi \in \Sigma$ is valid over the class of Σ -quasimodels. The converse is true, and for this we need to construct, given a dynamical model \mathfrak{X} falsifying φ , a quasimodel also falsifying φ . This quasimodel will be denoted \mathfrak{X}/Σ . The worlds of \mathfrak{X}/Σ will be called Σ -moments; the intuition is that a Σ -moment represents all the information that holds at the same moment of time.

Definition 7.1 (moment). *A Σ -moment is a Σ -labeled space \mathfrak{w} such that $\mathcal{T}_{\mathfrak{w}}$ is the down-set topology of a partial order $\preceq_{\mathfrak{w}}$, and $(|\mathfrak{w}|, \preceq_{\mathfrak{w}})$ is a finite tree with (unique) root $r_{\mathfrak{w}}$. We will write $\ell_{\Sigma}(\mathfrak{w})$ instead of $\ell_{\mathfrak{w}}(r_{\mathfrak{w}})$. The set of Σ -moments is denoted M_{Σ} .*

Definition 7.2 (submoment). *Let \mathfrak{w} be a Σ -moment. For $w \in |\mathfrak{w}|$, let $\mathfrak{w}[w] = \langle \downarrow w, \preceq_{\mathfrak{w}} \upharpoonright \downarrow w, \ell_{\mathfrak{w}} \upharpoonright \downarrow w \rangle$. We write $\mathfrak{w} \preceq_{\Sigma} \mathfrak{v}$ if $\mathfrak{v} = \mathfrak{w}[w]$ for some $w \in |\mathfrak{w}|$.*

We now wish to define a weak Σ -quasimodel \mathbb{M}_{Σ} over M_{Σ} . For this, it remains to define a sensible relation on M_{Σ} .

Definition 7.3 (temporal successor). *Say \mathfrak{w} is a temporal successor of \mathfrak{v} , denoted $\mathfrak{v} S_{\Sigma} \mathfrak{w}$, if there exists a sensible relation $R \subseteq |\mathfrak{v}| \times |\mathfrak{w}|$ such that $r_{\mathfrak{v}} R r_{\mathfrak{w}}$.*

Lemma 7.4. *Let $\Sigma \in \mathcal{L}$ and $\mathfrak{w}, \mathfrak{w}', \mathfrak{v} \in M_{\Sigma}$.*

1. *If $\mathfrak{w} S_{\Sigma} \mathfrak{v}$ then $(\ell_{\Sigma}(\mathfrak{w}), \ell_{\Sigma}(\mathfrak{v}))$ is sensible.*
2. *If $\mathfrak{w}' \preceq_{\Sigma} \mathfrak{w} S_{\Sigma} \mathfrak{v}$, there is $\mathfrak{v}' \preceq_{\Sigma} \mathfrak{v}$ such that $\mathfrak{w}' S_{\Sigma} \mathfrak{v}'$.*

Proof. Suppose that $\mathfrak{w} S_{\Sigma} \mathfrak{v}$, and let $R \subseteq |\mathfrak{v}| \times |\mathfrak{w}|$ be a sensible relation such that $r_{\mathfrak{v}} R r_{\mathfrak{w}}$. Then, the pair $(\ell_{\mathfrak{w}}(r_{\mathfrak{w}}), \ell_{\mathfrak{v}}(r_{\mathfrak{v}}))$ is sensible because R is sensible, but it is equal to $(\ell_{\Sigma}(\mathfrak{w}), \ell_{\Sigma}(\mathfrak{v}))$, as needed for the first claim.

For the second, suppose further that $\mathfrak{w}' \preceq_{\Sigma} \mathfrak{w}$. This means that $\mathfrak{w}' = \mathfrak{w}[w]$ for some $w \in |\mathfrak{w}|$. Since R is sensible and $|\mathfrak{v}|$ is open, we have that $R^{-1}|\mathfrak{v}|$ is open as well, meaning in particular that $w R v$ for some $v \in |\mathfrak{v}|$. Now consider $R' = R \cap (\downarrow w \times \downarrow v)$. Since $\downarrow v$ is open it follows that R' is continuous as well, and every pair in R' is sensible since every pair in R was. Moreover, $r_{\mathfrak{w}[w]} = w R' v = r_{\mathfrak{v}[v]}$. Thus R' witnesses that $\mathfrak{w}' = \mathfrak{w}[w] S_{\Sigma} \mathfrak{v}[v] \preceq_{\Sigma} \mathfrak{v}$, as needed. \square

Now we are ready to define our ‘canonical’ weak quasimodels.

Definition 7.5 (\mathbb{M}_Σ). Let $\Sigma \in \mathcal{L}$. Define M_Σ to be the set of all finite Σ -moments, and set $\mathbb{M}_\Sigma = (M_\Sigma, \preceq_\Sigma, S_\Sigma, \ell_\Sigma)$.

Lemma 7.6. If $\Sigma \in \mathcal{L}$, then \mathbb{M}_Σ is a weak quasimodel.

Proof. That S_Σ is sensible is Lemma 7.4, so it remains to show that $(M_\Sigma, \preceq_\Sigma, \ell_\Sigma)$ is a Σ -labeled space. First we check that ℓ_Σ is continuous. This amounts to showing that, if $\mathfrak{w} \succ_\Sigma \mathfrak{v} \in M_\Sigma$, then $\ell_\Sigma(\mathfrak{w}) \subseteq \ell_\Sigma(\mathfrak{v})$. But, $\mathfrak{w} \succ_\Sigma \mathfrak{v}$ means that $\mathfrak{v} = \mathfrak{w}[v]$ for some $v \preceq_{\mathfrak{w}} r_{\mathfrak{w}}$, hence $\ell_\Sigma(\mathfrak{v}) = \ell_{\mathfrak{w}}(v) \supseteq \ell_{\mathfrak{w}}(r_{\mathfrak{w}}) = \ell_\Sigma(\mathfrak{w})$ by the continuity of $\ell_{\mathfrak{w}}$. Similarly, if $\delta = \varphi \rightarrow \psi \in \partial \ell_\Sigma(\mathfrak{w})$, then there is $v \preceq_{\mathfrak{w}} r_{\mathfrak{w}}$ such that v revokes δ ; but then, $\mathfrak{w}[v] \preceq_\Sigma \mathfrak{w}$ and $\mathfrak{w}[v]$ revokes δ . \square

Henceforth we may write \preceq, S, ℓ instead of $\preceq_\Sigma, S_\Sigma, \ell_\Sigma$ when this does not lead to confusion. Observe that S_Σ is not necessarily ω -sensible, and ℓ_Σ is not necessarily extensive, so \mathbb{M}_Σ is not a quasimodel as it stands. It does, however, contain substructures which are proper quasimodels, as we will see later.

7.1 Building moments from smaller moments

Often we will want to construct a Σ -moment from smaller moments. Here we will define the basic operation we will use to do this, and establish the conditions that the pieces must satisfy. Below, \coprod denotes a disjoint union.

Definition 7.7. Let φ be a formula of ITLC, $\Sigma \in \mathcal{L}$, $\Phi \in \mathbb{T}_\Sigma$ and $U \subseteq \mathbb{M}_\Sigma$. Define $\mathfrak{w} = \frac{U}{\Phi}$ by setting

$$\begin{aligned} |\mathfrak{w}| &= \{\Phi\} \cup \coprod_{u \in U} |u|, \\ \preceq_{\mathfrak{w}} &= (\{\Phi\} \times |\mathfrak{w}|) \cup \coprod_{u \in U} \preceq_u, \\ \ell_{\mathfrak{w}}(v) &= \{(\Phi, \Phi)\} \cup \coprod_{u \in U} \ell_u(v). \end{aligned}$$

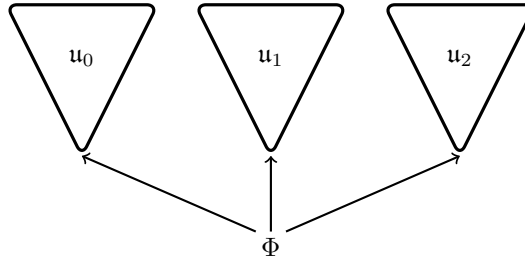


Figure 2: The moment $\frac{U}{\Phi}$, where $U = \{u_0, u_1, u_2\}$ and $\Phi \subseteq \Sigma \subseteq \mathcal{L}$.

Note that $\frac{U}{\Phi}$ will not always be a moment, since $\ell_{\frac{U}{\Phi}}$ thus defined might not be continuous, or it might not revoke some defect. To ensure that we do obtain a new moment, we need for (Φ, U) to be a kit.

Definition 7.8 (Σ -kit). Fix $\Sigma \in \mathcal{L}$.

1. A Σ -kit is a pair (Φ, U) , with $\Phi \in \mathbb{T}_\Sigma$ and $U \subseteq \mathbb{M}_\Sigma$ finite, such that
 - (a) $\Phi \subseteq \ell(\mathbf{u})$ for all $\mathbf{u} \in U$, and
 - (b) whenever $\varphi \rightarrow \psi \in \partial\Phi$, we have that there is $\mathbf{u} \in U$ such that $\varphi \rightarrow \psi \notin \ell(\mathbf{u})$.
2. If $\mathbf{w} \in \mathbb{M}_\Sigma$, we say that the Σ -kit (Φ, U) is a successor kit for \mathbf{w} if:
 - (a) the pair $(\ell(\mathbf{w}), \Phi)$ is temporally adequate, and
 - (b) if $\mathbf{v} \prec \mathbf{w}$, there is $\mathbf{u} \in U$ such that $\mathbf{v} S_\Sigma \mathbf{u}$.

Lemma 7.9. Let $\Sigma \in \mathcal{L}$, $\Phi \subseteq \Sigma$, and $V \subseteq \mathbb{M}_\Sigma$.

Then,

1. $\frac{V}{\Phi}$ is a Σ -moment if and only if (Φ, V) is a Σ -kit, and
2. $\frac{V}{\Phi}$ is a Σ -moment with $\mathbf{w} S_\Sigma \frac{V}{\Phi}$ whenever (Φ, V) is a successor kit for \mathbf{w} .

Proof. To prove 1, one can check that the conditions for being a Σ -kit correspond exactly to the conditions in Definition 5.1. For 2, let w_1, \dots, w_n enumerate the set $|\mathbf{w}| \setminus \{r_{\mathbf{w}}\}$, and for $i \leq n$, let $\mathbf{w}_i = \mathbf{w}[w_i]$. For $i \leq n$, let $\mathbf{v}_i \in V$ be such that $\mathbf{w}_i S_\Sigma \mathbf{v}_i$, and $R_i \subseteq |\mathbf{w}_i| \times |\mathbf{v}_i|$ be sensible. Then, define $R \subseteq |\mathbf{v}| \times |\frac{V}{\Phi}|$ by $w R v$ if and only if (a) $w = r_{\mathbf{w}}$ and $v = r_{\mathbf{v}}$, or (b) $w \in |\mathbf{w}_i|$, $v \in |\mathbf{v}_i|$ and $w R_i v$. It is readily seen that the relation R thus defined is sensible. \square

8 Simulations

In order to prove that every falsifiable formula is falsifiable in a quasimodel, we need a technique to represent dynamical models using quasimodels. The relation between a dynamical model and a quasimodel is described by simulations:

Definition 8.1 (simulation). Let $\Sigma \in \mathcal{L}$ and $\mathfrak{X}, \mathfrak{Y}$ be labeled spaces. A continuous relation $\chi \subseteq |\mathfrak{X}| \times |\mathfrak{Y}|$ is a simulation if, for all $(x, y) \in \chi$, $\ell_{\mathfrak{X}}(x) = \ell_{\mathfrak{Y}}(y)$.

We will call the latter property *label-preservation*. Note that it may be that $\ell_{\mathfrak{X}}$ is extensional while $\ell_{\mathfrak{Y}}$ is not, or vice-versa. However, this cannot happen with a total, surjective simulation:

Lemma 8.2. Suppose that $\Sigma \in \mathcal{L}$, $\mathfrak{X}, \mathfrak{Y}$ are labeled spaces, and $\chi \subseteq |\mathfrak{X}| \times |\mathfrak{Y}|$ is a total, surjective simulation. Then, $\ell_{\mathfrak{X}}$ is extensional if and only if $\ell_{\mathfrak{Y}}$ is extensional.

Simulations are useful for comparing the purely topological behavior of labeled structures. However, to compare the dynamic behavior, we need something a bit stronger.

Definition 8.3 (dynamic simulation). Let $\Sigma \in \mathcal{L}$ and $\mathfrak{X}, \mathfrak{Y}$ be labeled systems. A dynamic simulation between \mathfrak{X} and \mathfrak{Y} is a simulation $\chi \subseteq |\mathfrak{X}| \times |\mathfrak{Y}|$ such that $S_{\mathfrak{X}}\chi \subseteq \chi S_{\mathfrak{Y}}$.

The following properties are readily verified:

Lemma 8.4. *Let $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}$ be labeled systems and $\chi \subseteq |\mathfrak{X}| \times |\mathfrak{Y}|$, $\xi \subseteq |\mathfrak{Y}| \times |\mathfrak{Z}|$ be simulations. Then:*

1. $\xi\chi \subseteq |\mathfrak{X}| \times |\mathfrak{Z}|$ is a simulation. Moreover, if both χ and ξ are dynamic, then so is $\xi\chi$.
2. If $U \subseteq |\mathfrak{X}|$ and $V \subseteq |\mathfrak{Y}|$ are open, then $\chi \upharpoonright U \times V$ is a simulation.
3. If $\Xi \subseteq \mathcal{P}(|\mathfrak{X}| \times |\mathfrak{Y}|)$ is a set of simulations, then $\bigcup \Xi$ is also a simulation.

Suppose that χ is a dynamic simulation between a weak quasimodel \mathfrak{Q} and a dynamical model \mathfrak{X} . It may be that $S_{\mathfrak{Q}}$ is not ω -sensible; however, as we will see, we may use χ to extract an ω -sensible substructure from \mathfrak{Q} .

Lemma 8.5. *Let $\mathfrak{W}, \mathfrak{V}$ be labeled systems and $\chi \subseteq |\mathfrak{W}| \times |\mathfrak{V}|$ be a dynamical simulation. Then, if $S_{\mathfrak{V}}$ is ω -sensible, it follows that $S_{\mathfrak{W}} \upharpoonright \chi^{-1}(|\mathfrak{V}|)$ is also ω -sensible.*

Proof. Let $w \in \chi^{-1}(|\mathfrak{V}|)$ and $\diamond\varphi \in \ell_{\mathfrak{W}}(w)$. Since $w \in \chi^{-1}(|\mathfrak{V}|)$, we can choose $v_0 \in |\mathfrak{V}|$ such that $w \chi v_0$. Since $\diamond\varphi \in \ell_{\mathfrak{W}}(w)$, it follows that $\diamond\varphi \in \ell_{\mathfrak{V}}(v_0)$. Using the assumption that $S_{\mathfrak{V}}$ is ω -sensible, we may choose a sequence $v_0 S_{\mathfrak{V}} v_1 S_{\mathfrak{V}} v_2 S_{\mathfrak{V}} \dots S_{\mathfrak{V}} v_n$ such that $\varphi \in \ell_{\mathfrak{V}}(v_n)$. But, since χ is a dynamic simulation, we may recursively find $w = w_0 S_{\mathfrak{W}} w_1 S_{\mathfrak{W}} w_2 S_{\mathfrak{W}} \dots S_{\mathfrak{W}} w_n$ such that $w_i \chi v_i$ for each $i \leq n$. But then we have that $\varphi \in \ell_{\mathfrak{W}}(w_n)$, as needed. \square

Suppose in particular that \mathfrak{X} is a dynamical model and $x_* \in |\mathfrak{X}|$ falsifies $\varphi \in \Sigma$. Then, if \mathfrak{Q} is a weak Σ -quasimodel and $\chi \subseteq |\mathfrak{Q}| \times |\mathfrak{X}|$ is a surjective dynamical simulation such that $w_* \chi x_*$ for some $w_* \in |\mathfrak{Q}|$, it follows $\mathfrak{Q} \upharpoonright \chi^{-1}(|\mathfrak{X}|)$ is a quasimodel falsifying φ . Thus our strategy will be to show that there is a weak Σ -quasimodel \mathfrak{Q} such that, given *any* dynamical model \mathfrak{X} , there is a surjective dynamical simulation $\chi \subseteq |\mathfrak{Q}| \times |\mathfrak{X}|$. In principle we could take $\mathfrak{Q} = \mathbb{M}_{\Sigma}$, but since we wish to obtain finite quasimodels, it will be convenient to consider a finite substructure of \mathbb{M}_{Σ} . The elements of this structure will be the irreducible Σ -moments, as defined in the next section.

9 Irreducible moments

In order to obtain finite quasimodels, we will restrict \mathbb{M}_{Σ} to moments that are, in a sense, no ‘bigger’ than they need to be. To be precise, we want them to be minimal with respect to \preceq , which we define below (along with some other useful relations between moments).

Definition 9.1 (reduct). *Let $\Sigma \in \mathcal{L}$ and $\mathfrak{w}, \mathfrak{v}$ be Σ -moments. We write*

1. $\mathfrak{w} \cong \mathfrak{v}$ if $\mathfrak{w}, \mathfrak{v}$ are isomorphic.
2. $\mathfrak{w} \sqsubseteq \mathfrak{v}$ if $|\mathfrak{w}| \subseteq |\mathfrak{v}|$, $\preceq_{\mathfrak{w}} = \preceq_{\mathfrak{v}} \upharpoonright |\mathfrak{w}|$, and $\ell_{\mathfrak{w}} = \ell_{\mathfrak{v}} \upharpoonright |\mathfrak{w}|$.

3. $\mathfrak{w} \leq \mathfrak{v}$ if $\mathfrak{w} \sqsubseteq \mathfrak{v}$ and there is continuous $\pi: |\mathfrak{v}| \rightarrow |\mathfrak{w}|$ such that $\ell_{\mathfrak{v}}(v) = \ell_{\mathfrak{w}}(\pi(v))$ for all $v \in |\mathfrak{v}|$ and $\pi^2 = \pi$. We say that \mathfrak{w} is a *reduct* of \mathfrak{v} and π is a *reduction*.

Lemma 9.2. *Let $\Sigma \in \mathcal{L}$ and $\mathfrak{w}, \mathfrak{w}', \mathfrak{v}, \mathfrak{v}', \mathfrak{u} \in M_{\Sigma}$.*

1. *If $\mathfrak{w} \leq \mathfrak{v}$ then $\ell(\mathfrak{w}) = \ell(\mathfrak{v})$.*
2. *If $\mathfrak{w} \leq \mathfrak{v}$ and $w \in |\mathfrak{w}|$ then $\mathfrak{w}[w] \leq \mathfrak{v}[w]$.*
3. *If $\mathfrak{w} \leq \mathfrak{v} \leq \mathfrak{u}$ then $\mathfrak{w} \leq \mathfrak{u}$.*
4. *If $\mathfrak{w} S_{\Sigma} \mathfrak{v}$, $\mathfrak{w}' \leq \mathfrak{w}$ and $\mathfrak{v}' \leq \mathfrak{v}$, then $\mathfrak{w}' S_{\Sigma} \mathfrak{v}'$.*

Proof. Let $\pi: |\mathfrak{v}| \rightarrow |\mathfrak{w}|$ be a reduction. For Claim 1, observe that $r_{\mathfrak{w}} = \pi(r_{\mathfrak{v}})$, since if $v \in |\mathfrak{w}|$, from $v \preceq r_{\mathfrak{w}}$ we obtain $v = \pi(v) \preceq \pi(r_{\mathfrak{v}})$, so that $r_{\mathfrak{w}}, \pi(r_{\mathfrak{v}})$ are both the greatest element of $|\mathfrak{w}|$ and thus equal. It follows that $\ell(\mathfrak{w}) = \ell_{\mathfrak{w}}(\pi(r_{\mathfrak{v}})) = \ell_{\mathfrak{v}}(r_{\mathfrak{v}}) = \ell(\mathfrak{v})$.

For Claim 2, let $\pi: |\mathfrak{v}| \rightarrow |\mathfrak{w}|$ be a reduction. Then, $\pi \upharpoonright \downarrow w: \downarrow w \rightarrow \downarrow w$ is label-preserving, continuous and idempotent. It follows that $\mathfrak{w}[w] \leq \mathfrak{v}[w]$.

For Claim 3, let $\pi: |\mathfrak{v}| \rightarrow |\mathfrak{w}|$ and $\rho: |\mathfrak{u}| \rightarrow |\mathfrak{v}|$ be reductions. Then, $\pi \circ \rho$ is continuous, label-preserving, and if $w \in |\mathfrak{w}|$, $\pi \circ \rho(w) = \pi(w) = w$.

For Claim 4, let $S \subseteq |\mathfrak{w}| \times |\mathfrak{v}|$ be sensible, $\iota: |\mathfrak{w}'| \rightarrow |\mathfrak{w}|$ be the inclusion map, and $\pi: |\mathfrak{v}| \rightarrow |\mathfrak{v}'|$ be a reduction. Then, one readily checks that $\pi S \iota \subseteq |\mathfrak{w}'| \times |\mathfrak{v}'|$ is sensible. \square

Definition 9.3 (irreducible moment). *A moment \mathfrak{w} is irreducible if whenever $\mathfrak{w} \leq \mathfrak{v}$, it follows that $\mathfrak{w} = \mathfrak{v}$.*

Lemma 9.4. *Let $\Sigma \in \mathcal{L}$.*

1. *If \mathfrak{w} is any Σ -moment, there is $\mathfrak{w} \leq \mathfrak{v}$ such that \mathfrak{v} is irreducible.*
2. *If \mathfrak{w} is irreducible and $\mathfrak{v} \preceq \mathfrak{w}$, then \mathfrak{v} is irreducible.*
3. *If \mathfrak{w} is irreducible, $\mathfrak{u}, \mathfrak{v} \prec^1 \mathfrak{w}$ and $\mathfrak{u} \cong \mathfrak{v}$, then $\mathfrak{u} = \mathfrak{v}$.*

Proof. For Claim 1, choose $\mathfrak{v} \leq \mathfrak{w}$ such that $\#|\mathfrak{v}|$ is minimal. Then, if $\mathfrak{u} \leq \mathfrak{v}$, we have that $\#|\mathfrak{u}| \leq \#|\mathfrak{v}|$ and, by Lemma 9.2.3, $\mathfrak{u} \leq \mathfrak{w}$; hence by minimality, $\#|\mathfrak{u}| = \#|\mathfrak{v}|$, and thus $\mathfrak{u} = \mathfrak{v}$. Since \mathfrak{u} was arbitrary, \mathfrak{v} is irreducible.

For Claim 2, if $\mathfrak{v} \preceq \mathfrak{w}$, then $\mathfrak{v} = \mathfrak{w}[w]$ for some $w \in |\mathfrak{w}|$. Let $\pi: \downarrow_{\mathfrak{w}} w \rightarrow \downarrow_{\mathfrak{w}} w$ be a reduction, and define $\rho: |\mathfrak{w}| \rightarrow |\mathfrak{w}|$ by $\rho(v) = \pi(v)$ if $v \preceq w$, $\rho(v) = v$ otherwise. It is readily checked that ρ is a reduction, and since \mathfrak{w} is irreducible, it must be surjective. However, this is only possible if π was already surjective.

Finally, for Claim 3, assuming otherwise, let $h: |\mathfrak{u}| \rightarrow |\mathfrak{v}|$ be an isomorphism; then, define $\pi: |\mathfrak{w}| \rightarrow |\mathfrak{w}|$ by $\pi(v) = h(v)$ if $v \in |\mathfrak{u}|$, $\pi(v) = v$ otherwise. It is readily checked that π is a reduction. \square

Much like the set of moments forms a weak quasimodel \mathbb{M}_{Σ} , the set of irreducible moments forms a weak quasimodel.

Definition 9.5 (\mathbb{I}_Σ). Let $\Sigma \in \mathcal{L}$. Define I_Σ to be the set of all irreducible Σ -moments, and set $\mathbb{I}_\Sigma = \mathbb{M}_\Sigma \upharpoonright I_\Sigma$.

Let us write $\mathfrak{w} \leq_0 \mathfrak{v}$ if $\mathfrak{w} \leq \mathfrak{v}$ and \mathfrak{v} is irreducible. Then we have:

Lemma 9.6. Let $\Sigma \in \mathcal{L}$. Then,

1. \mathbb{I}_Σ is a weak Σ -quasimodel, and
2. $\leq_0 \subseteq |\mathbb{I}_\Sigma| \times |\mathbb{M}_\Sigma|$ is a surjective, dynamic simulation.

Proof. It is immediate from Lemma 9.4.2 that I_Σ is open, and thus \mathbb{I}_Σ is a weak quasimodel by Lemma 5.2. Surjectivity of \leq_0 is Lemma 9.4.1, label-preservation is Lemma 9.2.1, continuity is Lemma 9.2.2, and dynamicity is Lemma 9.2.4. \square

We wish to show that \mathbb{I}_Σ is always finite. For this, we will first show that irreducible frames cannot be too ‘tall’. To make this precise, let us define the *height* of \mathfrak{w} as the largest n so that there exists a chain $w_1 \preceq_{\mathfrak{w}} w_2 \dots \preceq_{\mathfrak{w}} w_n$ and write $n = \text{hgt}(\mathfrak{w})$.

Lemma 9.7. Let \mathfrak{w} be an irreducible Σ -moment and $w, v \in |\mathfrak{w}|$. Then, if $w \preceq v$ and $\ell(w) = \ell(v)$, it follows that $w = v$.

Proof. We proceed by induction on $\text{hgt}(\mathfrak{w})$: if $w \neq r_{\mathfrak{w}}$, then we can apply the induction hypothesis to $\mathfrak{w}[w]$. So, we may assume otherwise. We may also assume that $v \prec^1 r_{\mathfrak{w}}$; for, if $r_{\mathfrak{w}} = v$, there is nothing to prove, and if not, choose v' such that $r_{\mathfrak{w}} \succ^1 v' \succ v$; then, $\ell(r_{\mathfrak{w}}) \subseteq \ell(v') \subseteq \ell(v) = \ell(r_{\mathfrak{w}})$, so we also have $\ell(v') = \ell(r_{\mathfrak{w}})$. Then, define π by $\pi(v) = r_{\mathfrak{w}}$, $\pi(u) = u$ otherwise. It is readily seen that π is a reduction and $v \notin \pi|\mathfrak{w}|$, contradicting the irreducibility of \mathfrak{w} . \square

The following is then immediate from Lemma 9.7:

Corollary 9.8. If $\Sigma \in \mathcal{L}$ and \mathfrak{w} is an irreducible Σ -frame, then $\text{hgt}(\mathfrak{w}) \leq \#\Sigma + 1$.

Next we will give a bound on the size of irreducible frames. Our bound will be superexponential; recall that the superexponential 2_n^m is defined by recursion on n by $2_0^m = m$ and $2_{n+1}^m = 2^{2_n^m}$. We begin with a useful inequality:

Lemma 9.9. For all $m, n \geq 1$, $2^m \cdot 2_n^{(n-1)m} \leq 2_n^{nm}$.

Proof. Proceed by induction on n . If $n = 1$, then

$$2^m \cdot 2_1^{(1-1)m} = 2^m \cdot 1 = 2_1^{1m}.$$

If $n > 1$, then observe that $m < 2^m \leq 2_n^{(n-1)m}$. Then, $2^m \cdot 2_n^{(n-1)m} = 2^m \cdot 2^{2_n^{(n-1)m}} = 2^{m+2_n^{(n-1)m}} \leq 2^{2 \cdot 2_n^{(n-1)m}} \leq 2^{2^m \cdot 2_n^{(n-1)m}} \stackrel{\text{IH}}{\leq} 2^{2_n^{nm}} = 2_n^{nm}$, as needed. \square

Lemma 9.10. *If $\Sigma \in \mathcal{L}$ and $s = \#\Sigma > 0$, then*

$$\#I_\Sigma \leq 2^{s^2+s}_{s+1}.$$

Proof. We prove, by induction on n , that there are at most 2_n^{ns} irreducible moments with height n . Note that there are no moments of height 0, and $0 = 2_0^{0s}$. So, we may assume $n \geq 1$. Any irreducible of height at most n is of the form $\frac{A}{\Phi}$, where $\Phi \subseteq \Sigma$ and A is a set of distinct irreducibles of height less than n . There are at most 2^s choices for Φ . By induction hypothesis, there are at most $2_{n-1}^{(n-1)s}$ possible elements for A , and thus $2^{2_{n-1}^{(n-1)s}} = 2_n^{(n-1)s}$ choices for A . Hence there are at most $2^s \cdot 2_n^{(n-1)s} \leq 2_n^{ns}$ such \mathfrak{w} . The lemma then follows by choosing $n = s + 1$, the greatest value that $\text{hgt}(\mathfrak{w})$ could take. \square

Now that we know that \mathbb{I}_Σ is finite, it would be convenient if, whenever χ is a simulation and \mathfrak{w} is any Σ -moment such that $\mathfrak{w} \chi x$, we could replace \mathfrak{w} by some irreducible $\mathfrak{w}' \sqsubseteq \mathfrak{w}$ and still have that $\mathfrak{w}' \chi x$. The following operations on simulations will help us achieve this.

Definition 9.11. *Let $\Sigma \in \mathcal{L}$, \mathfrak{X} be a Σ -labeled space, and $\chi \subseteq M_\Sigma \times |\mathfrak{X}|$.*

1. *Define the reductive closure of χ by $\check{\chi} = \chi \sqsubseteq$. If $\check{\chi} = \chi$, we say χ is reductive.*
2. *Define the irreducible part of χ by $\chi_0 = \chi \upharpoonright I_\Sigma$.*

Let us see that these operations indeed produce new simulations.

Lemma 9.12. *Suppose that $\Sigma \in \mathcal{L}$, \mathfrak{X} is a dynamical model, and $\chi \subseteq M_\Sigma \times |\mathfrak{X}|$ is a simulation. Then,*

1. *$\check{\chi}$ and χ_0 are also simulations.*
2. *If χ is reductive, then*
 - (a) *$\chi_0(I_\Sigma) = \chi(M_\Sigma)$, and*
 - (b) *if $\mathfrak{w} \chi x$, $\mathfrak{v} \chi f_{\mathfrak{X}}(x)$, and $\mathfrak{w} S_\Sigma \mathfrak{v}$, there is \mathfrak{v}' such that $\mathfrak{v}' \chi_0 f_{\mathfrak{X}}(x)$ and $\mathfrak{w} S_\Sigma \mathfrak{v}'$.*

Proof. That $\check{\chi}$ is a simulation is immediate from Lemma 8.4.1, and it is easily seen that χ_0 is a simulation using Lemma 8.4.2. For claim 2a, if χ is reductive, then it is obvious that $\chi_0(I_\Sigma) \subseteq \chi(M_\Sigma)$. For the other inclusion, consider $x \in \chi(M_\Sigma)$, so that there is $\mathfrak{w} \in M_\Sigma$ with $\mathfrak{w} \chi x$. But by Lemma 9.6, \sqsubseteq_0 is surjective, so we can pick \mathfrak{v} with $\mathfrak{v} \sqsubseteq_0 \mathfrak{w}$; since χ is reductive, it follows that $\mathfrak{v} \chi x$, and hence $x \in \chi_0(I_\Sigma)$, as needed. Claim 2b is proven similarly, using Lemma 9.2.4. \square

10 Simulating dynamical systems

Our work with irreducibles shows that, if we can construct surjective, dynamic simulations on \mathbb{M}_Σ , then we immediately get surjective, dynamic simulations on \mathbb{I}_Σ , which we can then use to construct quasimodels. In this section, we will show how such simulations can be constructed.

Lemma 10.1. *Given any dynamical model \mathfrak{X} and $\Sigma \in \mathcal{L}$, there exists a greatest simulation $\chi^* \subseteq M_\Sigma \times |\mathfrak{X}|$. Moreover, χ^* is reductive.*

Proof. By Lemma 8.4.3,

$$\chi^* = \bigcup \{ \chi \subseteq I_\Sigma \times |\mathfrak{X}| : \chi \text{ is a simulation} \}$$

is a simulation, and clearly this is the greatest simulation. By Lemma 9.12, $\tilde{\chi}^*$ is also a simulation, hence $\tilde{\chi}^* \subseteq \chi^*$ and thus χ^* is reductive. \square

Our goal is to prove that χ^* gives us a surjective, dynamic simulation. The following lemma will be essential in proving this.

Lemma 10.2 (simulation extensions). *Let $\Sigma \in \mathcal{L}$, \mathfrak{X} be a dynamical model, and $\chi \subseteq M_\Sigma \times |\mathfrak{X}|$ be a simulation.*

Suppose that $\Phi \subseteq \Sigma$ and $A \subseteq M_\Sigma$ are a Σ -kit (as in Definition 7.8) and there is $x_ \in |\mathfrak{X}|$ such that: 1. $\ell_{\mathfrak{X}}(x_*) = \Phi$, and 2. for all $\mathfrak{w} \in A$, $x_* \in \overline{\chi(\mathfrak{w})}$. Then, $\zeta = \chi \cup \{(\frac{A}{\Phi}, x_*)\}$ is also a simulation.*

Proof. Note, first, that ζ is label-preserving. We must prove it is also continuous.

Consider an arbitrary open set $U \subseteq |\mathfrak{X}|$; we must show that $\zeta^{-1}(U)$ is open. So, let $\mathfrak{w} \in \zeta^{-1}(U)$, and assume that $\mathfrak{v} \preceq \mathfrak{w}$; let us see that $\mathfrak{v} \in \zeta^{-1}(U)$. Obviously the latter holds if $\mathfrak{w} = \mathfrak{v}$, so we assume $\mathfrak{v} \prec \mathfrak{w}$. Pick $x \in U$ such that $\mathfrak{w} \zeta x$. If $x \neq x_*$ or $\mathfrak{w} \neq \frac{A}{\Phi}$, then we already have $\mathfrak{w} \chi x$, and hence $\mathfrak{v} \in \chi^{-1}(U) \subseteq \zeta^{-1}(U)$. So, assume that $x = x_*$ and $\mathfrak{w} = \frac{A}{\Phi}$. Then, there is $\mathfrak{v}' \in A$ such that $\mathfrak{v} \prec \mathfrak{v}' \preceq \mathfrak{w}$. By assumption $x_* \in \overline{\chi(\mathfrak{v})}$, hence, since U is open, there is $y \in U$ such that $\mathfrak{v}' \chi y$. But χ is continuous and U is a neighborhood of y , so there is $z \in U$ such that $\mathfrak{v} \chi z$, hence $\mathfrak{v} \zeta z$ as needed. \square

Proposition 10.3. *For any dynamical model \mathfrak{X} and $\Sigma \in \mathcal{L}$, $\chi^* \subseteq M_\Sigma \times |\mathfrak{X}|$ is surjective.*

Proof. Since we know that χ^* is reductive, it suffices to show that a reductive simulation that is not surjective is not maximal. Thus, let $\chi \subseteq M_\Sigma \times |\mathfrak{X}|$ be any reductive simulation. Say a point $x \in X$ is *bad* if it does not lie in the range of χ . We claim that χ is not maximal if there are bad points.

If there are bad points, let $x_* \in |\mathfrak{X}|$ be a bad point such that $\ell(x_*)$ is \subseteq -maximal among all bad points. Let U_* be a neighborhood of x_* minimizing $\#\chi_0^{-1}(U_*)$, and such that $U_* \subseteq \llbracket \varphi \rrbracket_{\mathfrak{X}}$ for all $\varphi \in \ell(x_*)$. We claim that for each defect $\delta = \varphi \rightarrow \psi \in \partial\Phi$, there is $\mathfrak{u}_\delta \in \chi_0^{-1}(U_*)$ revoking δ . To see this, note that by the semantics of \rightarrow , there is $x_\delta \in U_*$ such that $x_\delta \in \llbracket \varphi \rrbracket_{\mathfrak{X}} \setminus \llbracket \psi \rrbracket_{\mathfrak{X}}$. But,

since $x_\delta \in U_*$ it follows that $\ell(x_\delta) \supseteq \ell(x_*)$, and since $\varphi \in \ell(x_\delta)$, the inclusion is strict. By maximality of $\ell(x_*)$ we have that x_δ is not bad, hence $x_\delta \in \chi(\mathfrak{v})$ for some $\mathfrak{v} \in M_\Sigma$. By Lemma 9.4, there is an irreducible $\mathfrak{u}_\delta \leq \mathfrak{v}$, and since χ is reductive, $\mathfrak{u}_\delta \chi x_\delta$, so that $\mathfrak{u}_\delta \in \chi_0^{-1}(U_*)$, as desired.

Let A be the set of all \mathfrak{u}_δ such that δ is a defect of $\ell(x_*)$. By Lemma 7.9, $\mathfrak{w}_* = \frac{A}{\ell(x_*)}$ is a Σ -moment, and we can set $\zeta = \chi \cup \{(\frac{A}{\ell(x_*)}, x_*)\}$. By Lemma 10.2, ζ is a simulation. Since x_* was bad, $\chi \subsetneq \zeta$, as desired. \square

Proposition 10.4. *Given a dynamical model \mathfrak{X} , χ^* is a dynamical simulation.*

Proof. The proof follows much the same structure as that of Proposition 10.3. Let χ be a reductive simulation, which in view of Proposition 10.3, we may assume to be surjective. Suppose further that χ is not dynamic; we will show that it cannot be maximal.

Say that x fails for \mathfrak{w} if $\mathfrak{w} \chi x$ but there is no $\mathfrak{v} \in \chi^{-1}f_{\mathfrak{X}}(x)$ such that $\mathfrak{w} S_\Sigma \mathfrak{v}$. We will show that χ is not maximal if any point fails for any moment. If this were the case, pick \mathfrak{w}_* of minimal height such that some point x_* fails for it. Let V_* be a neighborhood of $f_{\mathfrak{X}}(x_*)$ such that $\#\chi_0^{-1}(V_*)$ is minimal, and $V_* \subseteq \llbracket \varphi \rrbracket_{\mathfrak{X}}$ whenever $\varphi \in \ell(f_{\mathfrak{X}}(x_*))$.

As before, for each defect δ of $\ell(f_{\mathfrak{X}}(x_*))$, choose $\mathfrak{u}_\delta \in \chi_0^{-1}(V_*)$ revoking δ . Next, for each \mathfrak{v} such that $\mathfrak{v} \prec \mathfrak{w}$, we claim there is $\mathfrak{v}' \in \chi_0^{-1}(V_*)$ such that $\mathfrak{v} S_\Sigma \mathfrak{v}'$. To see this, using the continuity of $f_{\mathfrak{X}}$, let U be a neighborhood of x_* such that $f_{\mathfrak{X}}(U) \subseteq V_*$. Since χ is continuous, there is $x_{\mathfrak{v}} \in U$ such that $\mathfrak{v} \chi x_{\mathfrak{v}}$. Since $\text{hgt}(\mathfrak{v}) < \text{hgt}(\mathfrak{w})$, by the induction hypothesis, there is \mathfrak{v}'' such that $\mathfrak{v} S_\Sigma \mathfrak{v}''$ and $\mathfrak{v}'' \chi f_{\mathfrak{X}}(x_{\mathfrak{v}})$. But χ is reductive, so that by Lemma 9.12.2b, there is \mathfrak{v}' such that $\mathfrak{v}' \chi_0 f_{\mathfrak{X}}(x_{\mathfrak{v}}) \in V_*$ and $\mathfrak{v} S_\Sigma \mathfrak{v}'$, as needed.

Let

$$B = \{\mathfrak{u}_\delta : \delta \in \partial\ell(f_{\mathfrak{X}}(x_*))\} \cup \{\mathfrak{v}' : \mathfrak{v} \prec \mathfrak{w}\}.$$

Then, by Lemma 7.9, $\mathfrak{v}_* = \frac{B}{\ell(f_{\mathfrak{X}}(x_*))}$ is a Σ -moment and $\mathfrak{w}_* S_\Sigma \mathfrak{v}_*$ by Lemma 7.9.2.

We can then set $\zeta = \chi \cup \{(\mathfrak{v}_*, f_{\mathfrak{X}}(x_*))\}$. As before, ζ is a simulation which properly contains χ . Therefore, χ is not maximal. \square

We are now ready to prove our main result.

Definition 10.5. *Given a dynamical model \mathfrak{X} and $\Sigma \in \mathcal{L}$, define $\mathfrak{X}/\Sigma = \mathbb{I}_\Sigma \upharpoonright \text{dom}(\chi^*)$.*

Theorem 10.6. *If \mathfrak{X} is a dynamical model falsifying $\varphi \in \Sigma$, then \mathfrak{X}/Σ is a Σ -quasimodel falsifying φ .*

Proof. By Proposition 10.4, χ^* is a dynamical simulation, so by Lemmas 8.2 and 8.5, \mathfrak{X}/φ is a Σ -quasimodel. Now, pick $x_* \in |\mathfrak{X}| \setminus \llbracket \varphi \rrbracket_{\mathfrak{X}}$. By Proposition 10.3, χ_0^* is surjective, so there exists $\mathfrak{w}_* \in I_\Sigma$ such that $x_* \in \chi^*(\mathfrak{w}_*)$; hence $\varphi \in \Sigma \setminus \ell(\mathfrak{w}_*)$. This shows that that \mathfrak{X}/Σ falsifies φ , as desired. \square

Theorem 10.7. *ITLC is decidable.*

Proof. Taking $\Sigma = \text{sub}(\varphi)$, we see by Theorems 6.9 and 10.6 that φ is falsifiable if and only if it is falsifiable on a Σ -quasimodel of the form $\Omega = \mathfrak{X}/\Sigma$, which is a substructure of \mathbb{I}_Σ . Since Ω has at most $2^{(\|\varphi\|+1)\|\varphi\|}$ worlds, it remains to search for a quasimodel satisfying this bound. \square

11 Concluding Remarks

We have presented ITLC, a variant of Kremer’s intuitionistic dynamic topological logic, and shown it to be decidable. It is interesting to note the contrast with DTL, which is undecidable. Note, however, that the decision procedure we have given is not elementary, since the size of \mathbb{I}_Σ grows superexponentially on $\|\varphi\|$. Nevertheless, there is no reason to assume that this procedure is optimal, and a sharp lower bound on the complexity of the validity problem in ITLC remains to be found. In addition, the methods we have used rely on quasimodel-search procedures, and do not yield an axiomatization for ITLC.

There are variations of DTL which are decidable, but often with rather high complexity, as is the case of DTL of minimal systems. It is an interesting open problem to check if their intuitionistic variants are also more feasible.

Finally, we leave open the question of the decidability of the full ITLC*, or even of Kremer’s logic over the $\{\circ, \Box\}$ -fragment.

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